Topological rewriting systems applied to standard bases and syntactic algebras

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JOHANNES KEPLER UNIVERSITÄT LINZ

Der Wissenschaftsfonds.

I. Motivations

- ▷ Confluence property, polynomial reduction and Gröbner bases
- ▷ Rewriting formal power series and standard bases

II. Topological rewriting systems

- ▷ Topological confluence property
- ▷ Standard bases and topological confluence

III. Reduction operators

- ▷ Lattice structure
- ▷ Lattice characterisation of topological confluence

IV. Duality and syntactic algebras

- ▷ Syntactic algebras
- ▷ A duality criterion

V. Conclusion and perspectives

I. MOTIVATIONS

Some algorithmic problems in algebra

- solve decision problems (*e.g.*, membership problem)
- compute homological invariants (*e.g.*, Tor, Ext groups)
- analysis of functional systems (*e.g.*, integrability conditions)

← **Classical techniques**

Constructive methods in algebra

- compute set of representatives for congruence classes
- construct free resolutions of modules
- elimination theory for systems of equations

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Approach: orientation of relations in a structure ➔ notion of normal form

example: chosen orientation in $\mathbb{K}[x, y]$ ➔ induced by $yx \to xy$

NF computation: $3\ yxx + xyx - xy \to 4\ xyx - xy \to 4\ xxy - xy$

Remark on the case $\mathbb{K}[x, y]$ **:** NF monomials $x^n y^m$ form a linear basis

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MOTIVATING PROBLEM

Given an algebra $\mathbf{A} := \mathbb{K}\langle X \mid R\rangle$ presented by generators X and relations R

$$\mathbf{A} := \mathbb{K}\langle X\rangle / I(R) \qquad \Big(e.g., \quad \mathbb{K}[x,y] = \mathbb{K}\langle x, y \mid yx - xy\rangle\Big)$$

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Equivalently

do NF monomials form a generating family?

do NF monomials form a free family?

NF monomials do not form a generating family

$\mathbf{A} := \mathbb{K}\langle x \mid x - xx \rangle$ orientation: $x \to xx$

➔ $\dim_{\mathbb{K}}(\mathbf{A}) = 2$ ($\overline{1}$ and $\overline{x}$ form a basis)

➔ 1 is the only NF monomial ($\forall n \geq 1 : \quad x^n \to x^{n+1}$)

Definition: $\to$ is called terminating if

$\nexists$ infinite rewriting sequence

$$f_1 \to f_2 \to \cdots \to f_n \to f_{n+1} \to \dots$$

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Prop: let $\mathbf{A} := \mathbb{K}\langle X \mid R \rangle$. If $\to$ is a terminating orientation, then

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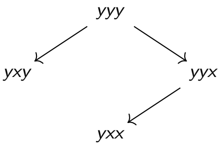

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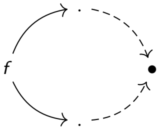

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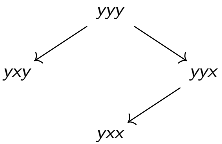

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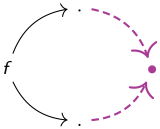

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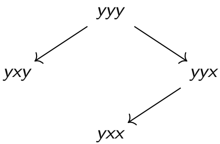

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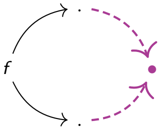

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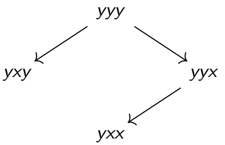

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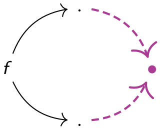

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Monomial orders

Well-founded total orders on X^*, product compatible

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Induces for $\mathbf{A} := \mathbb{K}\langle X \mid R\rangle$

Natural orientation

$\forall f = \mathrm{lc}(f)\,\mathrm{lm}(f) - \mathrm{rem}(f) \in R$

$\mathrm{lm}(f) \to_R 1/\,\mathrm{lc}(f)\,\mathrm{rem}(f)$

Gröbner bases definition

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Relationship

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Theorem. Let I be a (non)commutative polynomial ideal, R be a generating set of I, and $<$ be a monomial order. Then

$$R \text{ is a Gröbner basis of } I \quad \Leftrightarrow \quad \to_R \text{ is a confluent orientation}$$

Two applications of:

"Gröbner bases $\leftrightarrow$ confluent orientations"

Ideal membership problem: given a G.B. R of I and $f \in \mathbb{K}\langle X\rangle$, how to decide $f \in I$?

➔ reduce f into normal form $\widehat{f}$ using R and test $\widehat{f} = 0$

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PBW theorem: let $\mathscr{L}$ be a Lie algebra and let X be a totally well-ordered basis of $\mathscr{L}$.

Then, the universal enveloping algebra $U(\mathscr{L})$ of $\mathscr{L}$ admits as a basis

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Ideas of the proof:

- ➔ presentation of $U(\mathscr{L})$: $\mathbb{K}\langle X \mid yx - xy - [y,x], \quad x \neq y \in X\rangle$
- ➔ choice of terminating orientation: $yx \to xy + [y,x]$, where $x < y$
- ➔ this orientation is confluent (equivalent to Jacobi identity)
- ➔ a basis of $U(\mathscr{L})$ is composed of NF monomials: $x_1^{\alpha_1}\dots x_k^{\alpha_k}$ s.t. $x_i < x_{i+1}$

Monomial orders for formal power series

Definition: formal power series are linear maps $S : \mathbb{K}\langle X\rangle \to \mathbb{K}$, denoted by

$$S = \sum_{w \in X^*} (S, w)w$$

Leading monomials: selected w.r.t. the opposite order of a monomial order

➔ *e.g.*, $\mathrm{lm}\left(x + x^2 + x^3 + \dots\right) = x$

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G.B. def.: $\mathrm{lm}(I) = \langle \mathrm{lm}(G) \rangle$

Rewriting characterisation:

$\to_G$ is a confluent orientation

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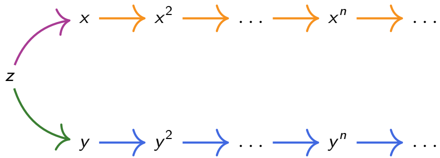
Rewriting characterisation: **?????**

Standard bases do not induce confluent rewriting systems

Example of standard basis: $X := \{z < y < x\}$ and I is generated by the standard basis

$$S := \left\{ \mathbf{z\text{-}x} \qquad \mathbf{z\text{-}y} \qquad \mathbf{x\text{-}x^2} \qquad \mathbf{y\text{-}y^2} \right\}$$

A non confluent diagram:



Fact: the two rewriting paths converge to 0 for the X-adic topology

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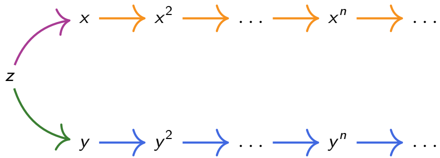

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OBJECTIVE OF THE TALK:

obtain a rewriting characterisation of standard bases

using a topological adaptation of the confluence property

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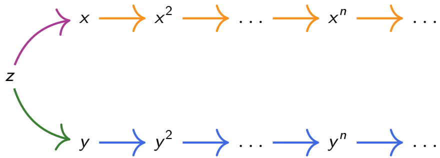

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II. TOPOLOGICAL REWRITING SYSTEMS

Objective: introduce a rewriting framework

that takes topology into account

Definition: a topological rewriting system $(A, \to, \tau)$ is given by a set A equipped with a binary relation $\to$ and a topology τ

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UNDERLYING IDEAS

The set A: set of syntactic expressions (polynomials, formal power series, λ/Σ-terms, ...)

The binary relation $\to$: represents rewriting steps

The topology τ: used to formalize the ideas "asymptotic rewriting and asymptotic confluence"

Asymptotic rewriting sequences

Let $(A, \to, \tau)$ be a topological rewriting system

Idea: a asymptotically rewrites into b if **a rewrites arbitrarily close to b**

Formally: we define $\twoheadrightarrow$ as being the $\tau_A^{\text{dis}} \times \tau$-closure of $\to$, *i.e.*

$$\mathrm{a} \twoheadrightarrow \mathrm{b} \qquad \text{iff} \qquad \Big(\forall\ \mathrm{U(b)} : \quad \exists . \in U(b), \quad a \xrightarrow{*} . \Big)$$

Asymptotic rewriting sequences

Let $(A, \rightarrow, \tau)$ be a topological rewriting system

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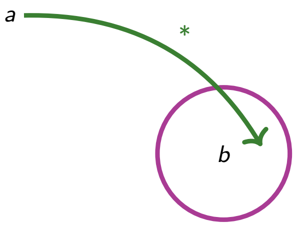
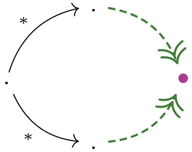
Pictorially:



Topological confluence property

Let $(A, \to, \tau)$ be a topological rewriting system

Definition: $\to$ is **τ-confluent** if divergent reductions asymptotically converge



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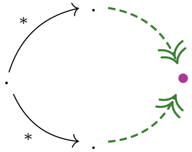
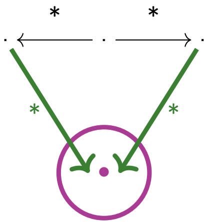


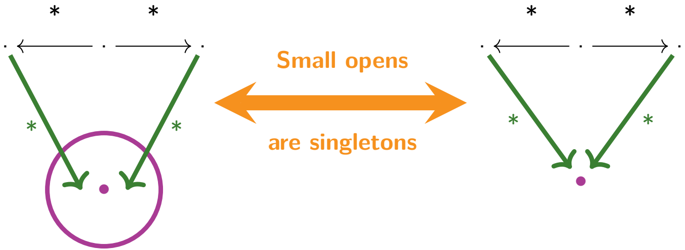
Alternatively: for **every neighbourhood of •**, there are **rewriting sequences** s.t.



Link with abstract rewriting theory

Abstract rewriting systems: $(A, \to, \tau)$, where $\tau := \tau_A^{\text{dis}}$ is the discrete topology

➔ asymptotic rewriting brings nothing new, *e.g.* τ-confluence $\Leftrightarrow$ confluence



Algebraic examples: word/polynomial/operadic/... rewriting

X-adic topology

on formal power series

Distance between FPSs: the distance between $S, S' \in \mathbb{K}\langle\langle X \rangle\rangle$ is defined by

$$d(S, S') := \frac{1}{2^{v(S-S')}}, \qquad \text{where} \quad v(S) := \min\left(\deg(w) \mid \quad (S, w) \neq 0\right)$$

➔ **"close series coincide until high degrees"**

Definition: the X-adic topology is the topology τ_X on $\mathbb{K}\langle\langle X \rangle\rangle$ induced by d

X-adic topology on formal power series

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Definition: the ***X*-adic topology** is the topology τ_X on $\mathbb{K}\langle\langle X\rangle\rangle$ induced by d

Theorem [C. 2020]

Let I be a formal power series ideal, S be a subset of I, and $<$ be a monomial order.

We have the following equivalence:

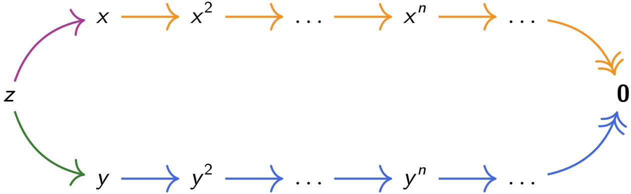
S is a standard basis of I $\quad\Leftrightarrow\quad$ S is a generating set of I and $\to_S$ is τ_X-confluent

Illustration of the theorem

Example: consider $X := \{z < y < x\}$ and I is generated by the standard basis

$$S := \{\mathbf{z\text{-}x} \quad \mathbf{z\text{-}y} \quad \mathbf{x\text{-}x^2} \quad \mathbf{y\text{-}y^2}\}$$

Rewriting diagram: we have the following τ_X-confluent diagram



Argument: the sequences $(x^n)_n, (y^n)_n \subseteq \mathbb{K}\langle\langle X\rangle\rangle$ both converge to $\mathbf{0}$ since

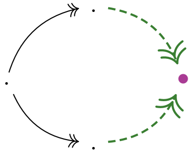
$$d(\mathbf{0}, x^n) = d(\mathbf{0}, y^n) = \frac{1}{2^n}$$

Some remarks

Theorem on standard bases: proven using a criterion of [Becker, 1990]

➔ criterion based on *S*-series (analogous to *S*-polynomials)

Alternative τ-confluence: diagram representation

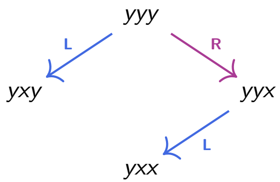


➔ appears in rewriting on infinitary λ/Σ-terms

III. REDUCTION OPERATORS

Functional representation of (discrete) rewriting systems

Example: $yy \to yx \quad \rightsquigarrow \quad$ **left**/**right** reduction operators on 3 letter words



Properties of L and R: they are linear projectors of $\mathbb{K}X^{(3)}$ (or $\mathbb{K}\langle X\rangle$) and compatible with the deglex order induced by $x < y$

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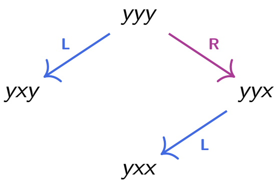

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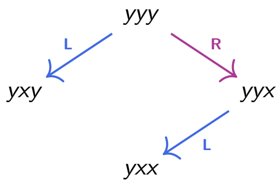


Properties of L and R: they are linear projectors of $\mathbb{K}X^{(3)}$ (or $\mathbb{K}\langle X\rangle$) and compatible with the deglex order induced by $x < y$

Definition: a **reduction operator** on a vector space V equipped with a well-ordered basis $(G, <)$ is a linear projector of V s.t.

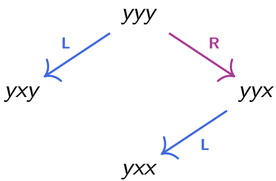
$$\forall g \in G : \qquad T(g) = g \quad \text{or} \quad \mathrm{lm}(T(g)) < g$$

Lattice structure

Proposition: the set of reduction operators admits lattice operations s.t.

$T_1 \wedge T_2$ computes minimal normal forms

Example: **L** $\wedge$ **R** maps 3-letter words starting with y to yxx



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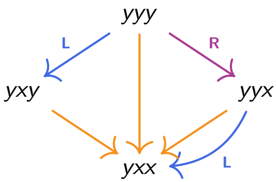

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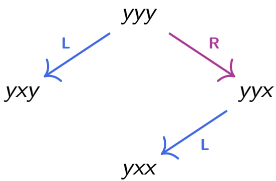

Functional characterisation of confluence (C. 2018):

the rewriting relation induced by T_1 and T_2 is confluent iff

$$\mathrm{im}(T_1) \cap \mathrm{im}(T_2) = \mathrm{im}(T_1 \wedge T_2)$$

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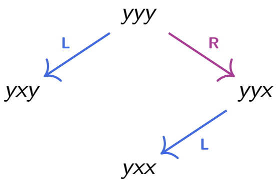

Illustration of the criterion:

➔ $yxy \in \mathrm{im}(\mathbf{L}) \cap \mathrm{im}(\mathbf{R})$

➔ $yxy \notin \mathrm{im}\left(\mathbf{L} \wedge \mathbf{R}\right)$

Functional characterisation of confluence (C. 2018):

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Objective: extend the functional approach to topological vector spaces

REQUIREMENTS

Compatibility with the topology
➔ continuous ROs

Extend the previous definition
➔ discrete topology

Motivating algebraic example
➔ formal power series

Locally well-ordered total bases

Fix a metric vector space (V, d) together with a subset $G \subset V$ s.t.

Totality: G is a free family that generates a dense subspace of V

Locally well-ordered: G is equipped with a total order $<$ and admits a strictly positive graduation $G = \coprod G^{(n)}$ s.t.

- ➔ $\forall g \in G^{(n)} : \quad 1/n \leq d(g, 0) < 1/(n-1)$
- ➔ $<$ restricts to well-orders on $G^{(n)}$'s

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Definition: a reduction operator is a continuous linear projector of V s.t.

$$\forall g \in G : \qquad T(g) = g \quad \text{or} \quad \mathrm{lm}(T(g)) < g$$

EXAMPLES OF LOCAL WELL-ORDERED TOTAL BASES

Discrete vector spaces

Metric: $\forall v \neq v' : \quad d(v, v') = 1$

➔ $G = G^{(1)}$ is a basis equipped with a total well-order

Remark: we recover the previous definition of reduction operator

Formal power series

Underlying space: $V = \mathbb{K}\langle\langle X \rangle\rangle$

Metric: X-adic metric

➔ $G = X^*$ is equipped with an opposite monomial order

➔ $G^{(2^n)} = \{\text{degree-}n \text{ monomials}\}$

Proposition: the kernel map induces a bijection between reduction operators on V and closed subspaces of V

$$\ker : \quad \left\{\textbf{reduction operators on } V\right\} \xrightarrow{\sim} \left\{\textbf{closed subspaces of } V\right\}$$

In particular, reduction operators admit the following lattice operations

- $T_1 \preceq T_2$ iff $\ker(T_2) \subseteq \ker(T_1)$
- $T_1 \wedge T_2$ is the reduction operator with kernel $\overline{\ker(T_1) + \ker(T_2)}$
- $T_1 \vee T_2$ is the reduction operator with kernel $\ker(T_1) \cap \ker(T_2)$

Theorem [C. 2020]

Let (V, d) be a metric vector space and let F be a set of reduction operators over V.

We have the following equivalence:

$$\to_F \quad \text{is topologically confluent} \quad \Leftrightarrow \quad \mathrm{im}\,(\wedge F) = \bigcap_{T \in F} \mathrm{im}(T)$$

IV. DUALITY AND SYNTACTIC ALGEBRAS

The functional approach brings DUALITY

$$\Big\{\textbf{reduction operators on } V\Big\} \quad \to \quad \Big\{\textbf{reduction operators on } V^*\Big\}$$

$$T \quad \mapsto \quad T^! \quad := \quad \mathrm{id}_{V^*} - T^*$$

➔ $\forall \varphi \in V^* : \quad T^!(\varphi) = \varphi - \varphi \circ T \in V^*$

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Some properties of the dual

Total basis of V^*: dual to the total basis of V (under some hypotheses)

➔ T^* is not a RO since $\left(\forall g \in G : \quad T^*(g^*) = g^* + \text{(other terms)}\right)$

Dual equations: $\mathrm{im}(T^!) = \mathrm{im}(T)^\perp \qquad \ker(T^!) = \ker(T)^\perp$

Duality and formal power series

Remark: from $\mathbb{K}\langle\langle X\rangle\rangle = (\mathbb{K}\langle X\rangle)^*$, there is a duality

$$\left\{\textbf{reduction operators on } \mathbb{K}\langle X\rangle\right\} \quad \to \quad \left\{\textbf{reduction operators on } \mathbb{K}\langle\langle X\rangle\rangle\right\}$$

Application: duality criterion for an algebra to be **syntactic** (next slides)

Definition: the **syntactic algebra** of $S \in \mathbb{K}\langle\langle X \rangle\rangle$ is

$$\mathbf{A}_S := \mathbb{K}\langle X \rangle / I_S$$

where I_S be the greatest ideal included in $\ker(S)$

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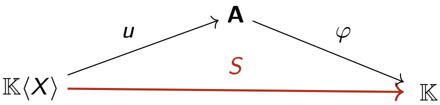
Syntactic algebras and series representations

Definition: a representation of S is a triple $\big(\mathbf{A},\ u : \mathbb{K}\langle X \rangle \to \mathbf{A},\ \varphi \in \mathbf{A}^*\big)$ s.t.



Fact: $\big(\mathbf{A}_S,\ \pi : \mathbb{K}\langle X \rangle \to \mathbf{A}_S,\ \overline{S} := S \mod I_S\big)$ is the minimal representation of S

➔ extension of Kleene's theorem: S is rational iff $\dim\big(\mathbf{A}_S\big) < \infty$ [Reutenaueur]

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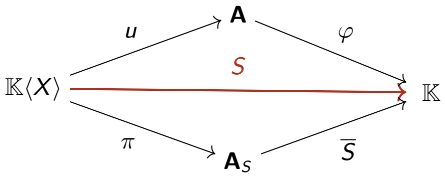

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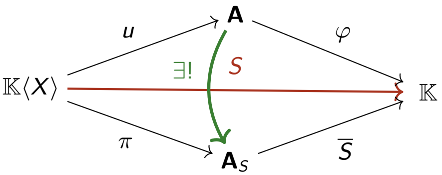

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Preliminaries

RO of an algebra: given a monomial order, $\mathbf{A} := \mathbb{K}\langle X\rangle/I$ is associated with

$$T_{\mathbf{A}} := \ker^{-1}(I): \qquad \text{reduction operator on } \mathbb{K}\langle X\rangle$$

Notation: given a reduction operator T, let $\widehat{\mathbb{K}\,\mathrm{im}(T)} \subseteq \mathbb{K}\langle\langle X\rangle\rangle$ defined by

$$S \in \widehat{\mathbb{K}\,\mathrm{im}(T)} \qquad \text{iff} \qquad \langle S \mid w\rangle \neq 0 \quad \Rightarrow \quad w \in \mathrm{im}(T)$$

Theorem [C. 2020]

Let $\mathbf{A} := \mathbb{K}\langle X\rangle/I$ be an algebra. Then, $\mathbf{A}$ is syntactic iff

$$\exists \text{ a nonzero } S \in \widehat{\mathbb{K}\,\mathrm{im}(T_{\mathbf{A}})} \quad \text{s.t.} \quad I \text{ is the greatest ideal included in } I \oplus \ker(S)$$

Moreover, in this case $\mathbf{A}$ is the syntactic algebra of $T^*(S) \in \mathbb{K}\langle\langle X\rangle\rangle$.

V. CONCLUSION AND PERSPECTIVES

Conclusion and perspectives

Summary of presented notions and results:

- ▷ we introduced the topological confluence property and a rewriting characterisation of standard bases
- ▷ we characterised topological confluence through lattice operations
- ▷ we formulated a duality criterion for an algebra to be syntactic

Further works:

- ▷ study abstract properties of topological rewriting systems (*e.g.*, C-R property, Newman's Lemma, etc ...)
- ▷ develop a geometrical framework for rewriting theory
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THANK YOU FOR LISTENING!