Reduction operators: completion, syzygies and Koszul duality

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Séminaire LIRICA

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I. Motivations

- computational problems and rewriting theory
- termination, confluence and Gröbner bases

II. Reduction operators

- reduction operators and linear rewriting systems
- lattice structure of reduction operators
- lattice descriptions of confluence and completion

III. Applications

- lattice structure and linear basis of syzygies
- construction of a contracting homotopy for the Koszul complex

IV. Conclusion

I. MOTIVATIONS

Rewriting theory and computational problems in algebra

Computational problems in algebra:

- how to compute linear bases for $\mathbb{K}$-algebras?
- solve decision problems, formal analysis of functional systems, computation of algebraic invariants, prove operator identities, ...

Rewriting theory: orientation of relations

- notion of normal forms $\rightsquigarrow$ "simple" representatives of congruence classes

Example: the polynomial algebra over two indeterminates

$\mathbb{K}[x, y] = \mathbb{K}\langle x, y \mid yx - xy\rangle$: noncommutative polynomials modulo $yx - xy \equiv 0$

- chosen orientation: $yx \to xy$
- a NF computation: $3\ yxx + xyx - xy \quad \to \quad 4\ xyx - xy \quad \to \quad 4\ xxy - xy$

In this case: NF monomials $x^n y^m$ form a basis of $\mathbb{K}[x, y]$

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Questions

Given $\mathbf{A} = \mathbb{K}\langle X \mid R\rangle$ presented by generators and oriented relations

do NF monomials form a linear basis of $\mathbf{A}$?

Equivalently:

- do NF monomials form a generating family of $\mathbf{A}$?
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Normalisation

Example: $\mathbb{K}\langle x \mid xx - x\rangle$ has basis $\{\overline{1}, \overline{x}\}$

- chosen orientation: $x \to xx \rightsquigarrow 1$ is the only NF monomial
- in general: NF monomials do not form a generating family

Definition: an orientation is terminating if there is no infinite rew. sequence

$$f_1 \to f_2 \to \cdots \to f_n \to f_{n+1} \to \cdots$$

Counterexample: $x \to xx$ and $f_n = x^n$

Fact

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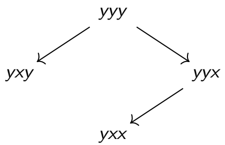

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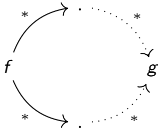

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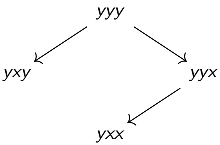

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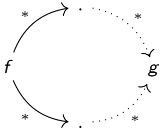

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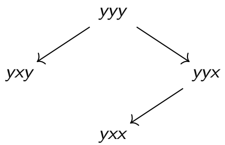

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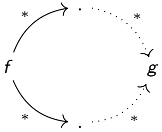

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Algebraic characterisations of confluence

Let I be a (non)commutative polynomial ideal, $R \subseteq I$ and $<$ a monomial order

Definition: R is a (non)commutative Gröbner basis of I if $\mathrm{lm}(R)$ generates $\mathrm{lm}(I)$

Rew. interpretation: $\{\mathrm{lm}(g) \to r(g) : g \in R\}$ is a confluent orientation

Illustration: $f \in I$ iff $f \xrightarrow{*}_R 0$ $\rightsquigarrow$ independent of the rew. path!

Reduction operators: representation theory of rew. systems

- formalisation of noncommutative GB [Bergman 78]
- lattice characterisation of quadratic GB applied to Koszul duality [Berger 98]

Objectives of the talk

Extend the functional approach

- lattice characterisation of the confluence property (for abstract linear rew. systems)
- lattice interpretation of completion
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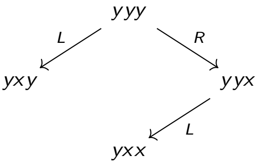
II. REDUCTION OPERATORS

Functional representations of rew. strategies

Example: $yy \to yx \rightsquigarrow$ **left**/**right**-reduction operators on 3 letter words



Properties of R.O.: L and R are functions that are

- endomorphisms of $G := \{3 \text{ letter words}\}$
- projectors, *i.e.*, $T^2 = T$
- not increasing w.r.t. $<_{\text{deglex}}$, *i.e.*,

$$\forall g \in G : \quad T(g) = g \quad \text{or} \quad T(g) <_{\text{deglex}} g$$

Remark: a S.R.S. can be embedded in a rew. system on noncommutative polynomials

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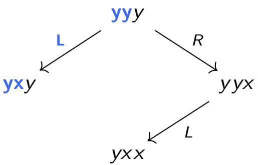

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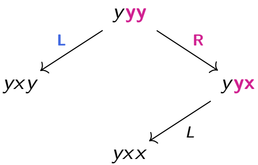

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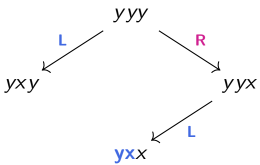

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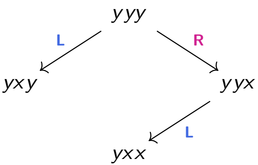

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Reduction operators

Fixed: a well-ordered set $(G, <)$, *e.g.*,

- noncommutative algebras: $G \rightsquigarrow$ words, $< \rightsquigarrow$ monomial order
- matrices: $G \rightsquigarrow$ a finite basis, $< \rightsquigarrow$ a rank on basis elements

Definition: a reduction operator relative to $(\mathbb{K}G, <)$ is a linear projector of $\mathbb{K}G$ s.t.

$$\forall g \in G : \quad T(g) = g \quad \text{or} \quad T(g) < g$$

Matrix representation for homogeneous algebras

For the rew. rule $yy \to yx$: L/R are left/right R.O. on

$$\mathbb{K}\{yxx, yxy, yyx, yyy\}$$

Matrix representation in the basis $yxx < yxy < yyx < yyy$:

$$L = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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Theorem

i. The kernel map induces a bijection between **RO** and subspaces of $\mathbb{K}G$:

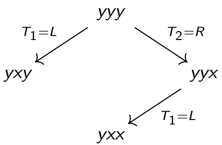
$$\ker : \mathbf{RO} \xrightarrow{\sim} \{\text{subspaces of } \mathbb{K}G\}, \quad T \mapsto \ker(T)$$

ii. **RO** admits lattice operations:

- $T_1 \preceq T_2$ iff $\ker(T_2) \subseteq \ker(T_1)$
- $T_1 \wedge T_2 := \ker^{-1}\left(\ker(T_1) + \ker(T_2)\right)$
- $T_1 \vee T_2 := \ker^{-1}\left(\ker(T_1) \cap \ker(T_2)\right)$

Fact

$T_1 \wedge T_2$ computes minimal normal forms, *e.g.*,



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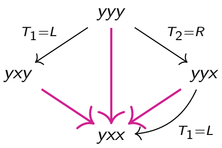

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- denote by $e_1 := yxx, e_2 := yxy, e_3 := yyx, e_4 := yyy$, so that

$$\ker(L) = \mathbb{K}\{\mathbf{e_3} - \mathbf{e_1}, \quad \mathbf{e_4} - \mathbf{e_2}\}, \qquad \ker(R) = \mathbb{K}\{\mathbf{e_4} - \mathbf{e_3}\}$$

- $\ker(L \wedge R) = \mathbb{K}\{e_3 - e_1, \quad e_4 - e_2, \quad e_4 - e_3\}$

  $\rightsquigarrow$ by Gaussian elimination

$$\ker(L \wedge R) = \mathbb{K}\{e_2 - e_1, \quad e_3 - e_1, \quad e_4 - e_1\}$$

  $\rightsquigarrow$ hence

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Lemma: $\forall T_1, T_2 \in \mathbf{RO}: \quad \mathrm{nf}(T_1 \wedge T_2) \subseteq \mathrm{nf}(T_1) \cap \mathrm{nf}(T_2)$

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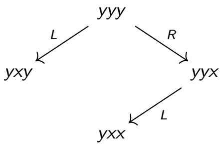

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We have the following lattice characterisation of confluence:

$$\rightarrow_F \text{ is confluent} \iff \mathrm{obs}(F) = \emptyset$$

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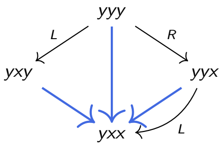

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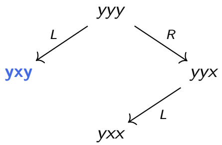

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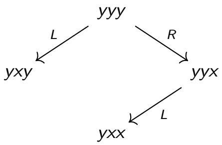

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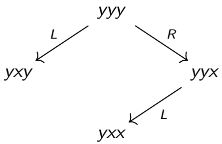

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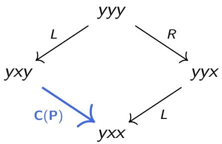

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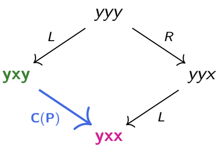

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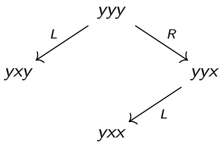

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III. APPLICATIONS

Problems involving syzygy computations:

Completion procedures: remove useless reductions/critical pairs

Higher-dimensional algebra: compute homological/homotopical invariants

Standardisation problems: choose a standard rew. path (*e.g.*, Janet bases)

Syzygies for R.O.

Fixed $F = \{T_1, \cdots, T_n\} \subseteq \mathbf{RO}$

Definition: the space of syzygies of F is the kernel of

$$\ker(T_1) \times \cdots \times \ker(T_n) \to \mathbb{K}G, \quad (v_1, \cdots, v_n) \mapsto v_1 + \cdots + v_n$$

Proposition: letting $F_i := \{T_1, \cdots, T_i\}$, there is a short exact sequence

$$0 \to \mathbf{syz}(F_{i-1}) \to \mathbf{syz}(F_i) \to \mathbf{syz}(\wedge F_{i-1}, T_i) \to 0$$

Moreover, $\forall T, T' \in \mathbf{RO}: \quad \mathbf{syz}(T, T') \simeq \ker(T \vee T')$

Consequence: a linear basis of $\mathbf{syz}(F)$ may by constructed by induction using

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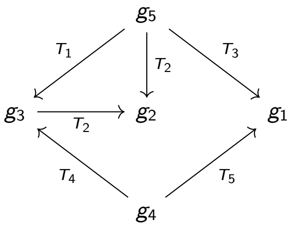

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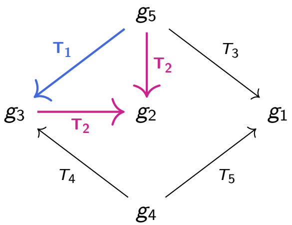

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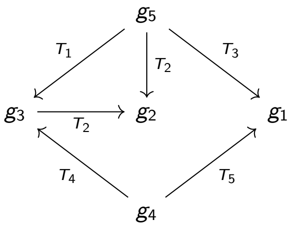

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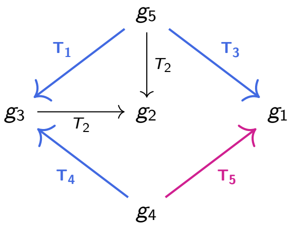

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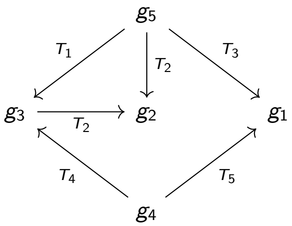

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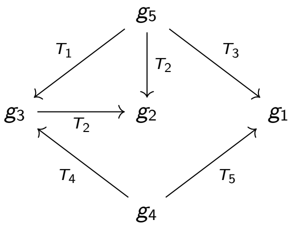

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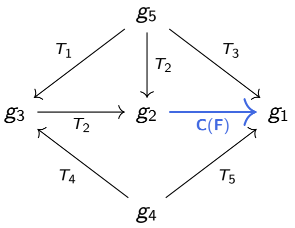

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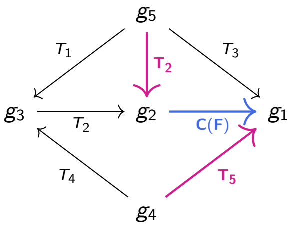

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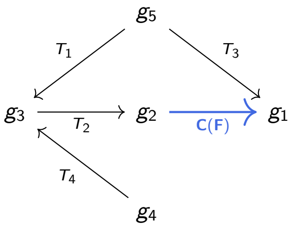

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Effective homological algebra

Free resolutions: consider an associative unital $\mathbb{K}$-algebra $\mathbf{A}$

- higher syzygies $\rightsquigarrow$ homological invariants of $\mathbf{A}$
- computing invariants requires to construct free resolutions, *i.e.*,

$$\cdots \xrightarrow{\partial_{n+1}} \mathbf{F}_n \xrightarrow{\partial_n} \mathbf{F}_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} \mathbf{F}_1 \xrightarrow{\partial_1} \mathbf{F}_0 \xrightarrow{\epsilon} \mathbb{K} \longrightarrow 0$$

where $\mathbf{F}_n$ are free modules and $\mathrm{im}(\partial_{n+1}) = \ker(\partial_n)$

Tke Koszul complex: assume $\mathbf{A}$ is homogeneous, *i.e.*, $\mathbf{A} = \mathbb{K}\langle X \mid R\rangle$, $R \subseteq \mathbb{K}X^{(N)}$

- a candidate: the Koszul complex $\rightsquigarrow$ $\mathbf{F}_n = \mathbf{A} \otimes J_n$, where

$$J_0 = \mathbb{K}, \quad J_1 = \mathbb{K}X, \quad J_2 = \mathbb{K}R, \quad J_3 = \Big(\mathbb{K}R \otimes \mathbb{K}X\Big) \cap \Big(\mathbb{K}X \otimes \mathbb{K}R\Big), \cdots$$

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A constructive proof of the Berger's criterion

Objective: constructive proof of Berger's criterion through a contracting homotopy

$$\rightsquigarrow \quad h_n : \mathbf{A} \otimes J_n \to \mathbf{A} \otimes J_{n+1} \quad \text{s.t.} \quad \partial_n h_{n+1} + h_n \partial_{n-1} = \mathrm{id}_{\mathbf{A} \otimes J_n}$$

$$\Rightarrow \qquad \ker(\partial_{n-1}) \quad = \quad \mathrm{im}(\partial_n)$$

Construction: given $\mathbf{A} = \mathbb{K}\langle X \mid R \rangle$ homogeneous and $<$ a monomial order

- $S := \ker^{-1}(\mathbb{K}R) \in \mathbf{RO}(X^*, <)$
- $T_1^n, T_2^n \quad \rightsquigarrow \quad$ formulas using S and lattice operations
- $h_n : \mathbf{A} \otimes J_n \to \mathbf{A} \otimes J_{n+1} \quad \rightsquigarrow \quad$ polynomial in (T_1^n, T_2^n)

The family $(h_n)_n$ is called the left bound of $\langle X \mid R \rangle$

Proposition: if $\langle X \mid R \rangle$ is side-confluent iff the reduction relations hold
Moreover, the extra-condition implies the reduction relations

Theorem

Let $\mathbf{A}$ be an homogeneous algebra satisfying the extra-condition and admitting a side-confluent presentation $\langle X \mid R \rangle$. Then, the left bound of $\langle X \mid R \rangle$ is a contracting homotopy for the Koszul complex of $\mathbf{A}$

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IV. CONCLUSION

Summary

Summary of presented results:

- lattice descriptions of confluence and completion
- lattice computation of syzygies
- construction of a contracting homotopy for the Koszul complex

Related results:

- lattice formulation of the noncommutative F_4 algorithm
- lattice classification of quotients of the magmatic operad
- representation theory of topological rew. systems applied to formal power series

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