Presenting isomorphic finitely presented modules by equivalent matrices: a constructive approach

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I. Motivations

- ▷ Equivalent linear functional systems
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- ▷ Fitting and Warfield's theorems

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- ▷ Morphisms of finitely presented modules
- ▷ Matrix characterisation of module isomorphisms

III. Constructive proofs of Fitting and Warfield's theorems

- ▷ Fitting's theorem
- ▷ Warfield's theorem

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I. MOTIVATIONS

Linear functional systems

Matrix representation: consider the linear system $R\eta = 0$, where

- ➔ $\eta = (\eta_1, \ldots, \eta_p)^T \in \mathcal{F}^p$: unknown vector in the functional space $\mathcal{F}$
- ➔ $R \in \mathbf{D}^{q \times p}$: matrix in the operator ring $\mathbf{D}$ ($\mathcal{F}$ is a left $\mathbf{D}$-module)

Example: consider the linear 3-order ODE

$$\dddot{\eta}(t) + a_2 \ddot{\eta}(t) + a_1 \dot{\eta} + a_0 \eta(t) = 0, \quad a_i \in \mathbb{Q}$$

- ➔ $\eta \in \mathcal{F} := C^\infty(I)$ and $R := (\partial^3 + a_2 \partial^2 + a_1 \partial + a_0) \in \mathbf{D} := \mathbb{Q}[\partial]$

Two notions of equivalent systems

Isomorphic solutions

Systems: $R\eta = 0$ $\quad R'\eta' = 0$

$(R \in \mathbf{D}^{q \times p} \quad R' \in \mathbf{D}^{q' \times p'})$

Equivalence: $\exists$ an isomorphism

$$\ker_{\mathcal{F}}(R.) \simeq \ker_{\mathcal{F}}(R'.)$$

Equivalent matrices

Systems: $L\eta = 0$ $\quad L'\eta' = 0$

$(L \in \mathbf{D}^{r \times c} \quad L' \in \mathbf{D}^{r \times c})$

Equivalence: $\exists X, Y$ invertible s.t.

$$LX = YL'$$

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$\ker_{\mathcal{F}}(R.) \simeq \ker_{\mathcal{F}}(R'.)$ **does not imply:**

R and R' are equivalent matrices

Counter-example: $\mathcal{F} := C^{\infty}(I)$, $\mathbf{D} := \mathbb{Q}[\partial]$, and

$$R\eta := (\partial^3 + a_2\partial^2 + a_1\partial + a_0)\eta = \dddot{\eta} + a_2\ddot{\eta} + a_1\dot{\eta} + a_0\eta = 0$$

$$R'\eta' := \begin{pmatrix} \partial & -1 & 0 \\ 0 & \partial & -1 \\ a_0 & a_1 & a_2 + \partial \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \dot{\eta}_0 - \eta_1 \\ \dot{\eta}_1 - \eta_2 \\ a_0\eta_0 + a_1\eta_1 + a_2\eta_2 + \dot{\eta}_2 \end{pmatrix} = 0$$

Then, $\ker_{\mathcal{F}}(R.) \simeq \ker_{\mathcal{F}}(R'.)$ is given by

$$\eta \mapsto \begin{pmatrix} \eta & \dot{\eta} & \ddot{\eta} \end{pmatrix}^T \quad \text{and} \quad \eta_0 \leftarrow\!\shortmid \begin{pmatrix} \eta_0 & \eta_1 & \eta_2 \end{pmatrix}^T$$

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A weaker result: R and R' may be extended into equivalent matrices

➔ requires suitable properties from **module theory**

Module of a linear functional system

Consider the linear system $R\eta = 0$, where $\eta \in \mathcal{F}^p$ and $R \in \mathbf{D}^{q\times p}$

Construction: $\mathbf{M} := \mathrm{coker}(.R) = \mathbf{D}^{1\times p} \,/\, \mathbf{D}^{1\times q}\, R$

➔ $\mathbf{M}$ is presented by p generators $y_1, \ldots, y_p$ submitted to q relations

$$\forall 1 \leq i \leq q : \qquad R_{i1}.y_1 + \cdots + R_{ip}.y_p \quad =_{\mathbf{M}} \quad 0$$

Remark: $\mathbf{M}$ is constructed such as the ring $\mathscr{O} := \mathbb{Q}[x_1, \ldots, x_p]/I(f_1, \ldots, f_q)$

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$$\mathbf{M} \simeq \mathbf{M}' \Rightarrow \left(\forall \mathcal{F} : \quad \ker_{\mathcal{F}}(R.) \simeq \ker_{\mathcal{F}}(R'.)\right)$$

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Objective: assuming $\mathbf{M} \simeq \mathbf{M}'$

➔ extend R and R' into equivalent matrices

Theoretical results: due to Fitting and Warfield

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Theorems [Fitting 1936, Warfield 1978].

Let $R \in \mathbf{D}^{q \times p}$ and $R' \in \mathbf{D}^{q' \times p'}$ be such that $\quad \mathbf{D}^{1 \times p} / \mathbf{D}^{1 \times q} R \simeq \mathbf{D}^{1 \times p'} / \mathbf{D}^{1 \times q'} R'$

Fitting's theorem: $L \in \mathbf{D}^{q+p'+(p+q) \times p+p'}$ and $L' \in \mathbf{D}^{(q+p')+p+q' \times p+p'}$ are equivalent, where

$$L := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \end{pmatrix} \leftarrow (p+q') \text{ rows} \qquad L' := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix} \leftarrow (q+p') \text{ rows}$$

Warfield's theorem: for $i, z \in \mathbb{N}$ small enough, $\tilde{L} \in \mathbf{D}^{q+(p'-i)+(p+q'-z) \times p+(p'-i)}$ and $\tilde{L}' \in \mathbf{D}^{(q+p'-z)+(p-i)+q' \times (p-i)+p'}$ are equivalent, where

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Objective of the talk

Construct invertible matrices $X, Y, \tilde{X}$, and $\tilde{Y}$ s.t.

$$LX = YL' \qquad \text{and} \qquad \tilde{L}\tilde{X} = \tilde{Y}\tilde{L}'$$

II. MODULE ISOMORPHISMS

Module presentations/morphisms

Let $R \in \mathbf{D}^{q \times p}$ and $\mathbf{M} := \mathbf{D}^{1 \times p} \,/\, \mathbf{D}^{1 \times q}\, R$

Elements of M: equivalence classes of rows $\lambda \in \mathbf{D}^{1 \times p}$ modulo the rows of R:

$$\forall 1 \leq i \leq q : \quad \big(0 \ldots \underbrace{1}_{i-th} \ldots 0\big) \begin{pmatrix} R_{11} & \ldots & R_{1p} \\ \vdots & \ldots & \vdots \\ R_{q1} & \ldots & R_{qp} \end{pmatrix} = (R_{i1} \quad \ldots \quad R_{ip})$$

➔ **M** has p generators $y_j := \Big[\big(0 \ldots 1 \ldots 0\big)\Big]$ and q relations

$$\Big[\,(R_{i1} \quad \ldots \quad R_{ip})\,\Big] =_{\mathbf{M}} R_{i1}.y_1 + \cdots + R_{ip}.y_p =_{\mathbf{M}} 0$$

Morphism $\mathbf{M} \xrightarrow{\varphi} \mathcal{F}$: uniquely determined by $\big(f_j := \varphi(y_j)\big) \in \mathcal{F}^p$ s.t.

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Malgrange's isomorphism:

$$\hom_{\mathbf{D}}(\mathbf{M}, \mathcal{F}) \quad \simeq \quad \ker_{\mathcal{F}}(R.)$$

Example

System: consider the linear system of PDEs

$$\begin{cases} \partial_1 \xi_1 =_{\mathcal{F}} 0 \\ \frac{1}{2}\left(\partial_2 \xi_1 + \partial_1 \xi_2\right) =_{\mathcal{F}} 0 \\ \partial_2 \xi_2 =_{\mathcal{F}} 0 \end{cases} \iff R\xi := \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 \\ 0 & \partial_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} =_{\mathcal{F}} 0$$

Corresponding module: $\mathbf{D} := \mathbb{Q}[\partial_1, \partial_2]$ and $\mathbf{M} := \mathbf{D}^{1\times 2} / \mathbf{D}^{1\times 3} R$

➔ $\mathbf{M}$ has 2 generators y_1, y_2 submitted to 3 relations

$$\partial_1 . y_1 =_{\mathbf{M}} 0, \qquad \frac{1}{2}\partial_2 . y_1 + \frac{1}{2}\partial_1 . y_2 =_{\mathbf{M}} 0, \qquad \partial_2 . y_2 =_{\mathbf{M}} 0$$

Malgrange's iso. illustration: let $\varphi : \mathbf{M} \to \mathcal{F}$ and $\xi_j := \varphi(y_j)$. Then, $R\xi =_{\mathcal{F}} 0$:

$$R\xi =_{\mathcal{F}} \begin{pmatrix} \partial_1 \xi_1 \\ \frac{1}{2}\partial_2 \xi_2 + \frac{1}{2}\partial_1 \xi_2 \\ \partial_2 \xi_2 \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} \varphi\left(\partial_1 . y_1\right) \\ \varphi\left(\frac{1}{2}\partial_2 . y_2 + \frac{1}{2}\partial_1 . y_2\right) \\ \varphi\left(\partial_2 . y_2\right) \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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Finite presentations and exact sequences

Consider a morphism of (finitely presented left $\mathbf{D}$-)modules

$$\varphi : \quad \mathbf{M} := \mathbf{D}^{1\times p} / \mathbf{D}^{1\times q} R \quad \to \quad \mathbf{M}' := \mathbf{D}^{1\times p'} / \mathbf{D}^{1\times q'} R'$$

Pullback of solutions: φ induces a morphism $\varphi^\star : \ker_{\mathcal{F}}(R'.) \to \ker_{\mathcal{F}}(R.)$, corresponding to

$$\varphi^\star : \hom_{\mathbf{D}}(\mathbf{M}', \mathcal{F}) \to \hom_{\mathbf{D}}(\mathbf{M}, \mathcal{F}), \quad \eta \mapsto \eta \circ \varphi$$

Question: what is the matrix representation of φ? ➔ requires exact sequences

Exact sequences: the presentation of $\mathbf{M}$ is equivalent to the following exact sequence

$$\mathbf{D}^{1\times q} \xrightarrow{.R} \mathbf{D}^{1\times p} \xrightarrow{\pi} \mathbf{M} \longrightarrow 0$$

i.e., $\left(.R : \mathbf{D}^{1\times q} \to \mathbf{D}^{1\times p}, \quad \mu \mapsto \mu R\right)$ and $\left(\pi : \mathbf{D}^{1\times p} \to \mathbf{M}, \quad \lambda \mapsto [\lambda]\right)$ are s.t.

$\left(\mathrm{im}(.R) = \ker(\pi)\right)$ and $\left(\mathrm{im}(\pi) = \ker(0) = \mathbf{M}, \quad \textit{i.e.} \quad \pi \text{ is onto}\right)$

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Proposition. $\left(\exists\, \varphi : \mathbf{M} \to \mathbf{M}'\right) \iff \left(\exists\, P \in \mathbf{D}^{p \times p'},\ Q \in \mathbf{D}^{q \times q'} : \quad RP = QR'\right)$

In this case, the following exact and commutative diagram holds

$$
\begin{array}{ccccccc}
\mathbf{D}^{1\times q} & \xrightarrow{.R} & \mathbf{D}^{1\times p} & \xrightarrow{\pi} & \mathbf{M} & \longrightarrow & 0 \\
\downarrow {\scriptstyle .Q} & & \downarrow {\scriptstyle .P} & & \downarrow {\scriptstyle \varphi} & & \\
\mathbf{D}^{1\times q'} & \xrightarrow{.R'} & \mathbf{D}^{1\times p'} & \xrightarrow{\pi'} & \mathbf{M}' & \longrightarrow & 0
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Proposition. $\left(\exists\, \varphi : \mathbf{M} \to \mathbf{M}'\right) \quad \Longleftrightarrow \quad \left(\exists\, P \in \mathbf{D}^{p\times p'},\ Q \in \mathbf{D}^{q\times q'} : \quad RP = QR'\right)$

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\mathbf{D}^{1\times q} & \xrightarrow{.R} & \mathbf{D}^{1\times p} & \xrightarrow{\pi} & \mathbf{M} & \longrightarrow & 0 \\
\downarrow .Q & & \downarrow .P & & \downarrow \varphi & & \\
\mathbf{D}^{1\times q'} & \xrightarrow{.R'} & \mathbf{D}^{1\times p'} & \xrightarrow{\pi'} & \mathbf{M}' & \longrightarrow & 0
\end{array}$$

Proof (sketch)

$\Longrightarrow$: φ corresponds to $\mathbf{m}' \in \ker_{\mathbf{M}'}(R.) \subseteq \mathbf{M}'^{p}$. Let $P \in \mathbf{D}^{p\times p'}$ defined by

$$\forall 1 \leq i \leq p : \quad \mathbf{m}'_i = P_{i1}.y'_1 + \cdots + P_{ip'}.y'_{p'}$$

$\mathbf{m}' \in \ker_{\mathbf{M}'}(R.)$ means: the rows of RP are combinations of rows of R', *i.e.*

$$\exists Q \in \mathbf{D}^{q\times q'} : \quad RP = QR'$$

Proposition. $\big(\exists\ \varphi : \mathbf{M} \to \mathbf{M}'\big) \quad \Longleftrightarrow \quad \big(\exists\ P \in \mathbf{D}^{p\times p'},\ Q \in \mathbf{D}^{q\times q'} : \quad RP = QR'\big)$

In this case, the following exact and commutative diagram holds

$$\begin{array}{ccccccc}
\mathbf{D}^{1\times q} & \xrightarrow{.R} & \mathbf{D}^{1\times p} & \xrightarrow{\pi} & \mathbf{M} & \longrightarrow & 0 \\
\downarrow {\scriptstyle .Q} & & \downarrow {\scriptstyle .P} & & \downarrow {\scriptstyle \varphi} & & \\
\mathbf{D}^{1\times q'} & \xrightarrow{.R'} & \mathbf{D}^{1\times p'} & \xrightarrow{\pi'} & \mathbf{M}' & \longrightarrow & 0
\end{array}$$

Proof (sketch)

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$$\exists Q \in \mathbf{D}^{q\times q'} : \quad RP = QR'$$

$\Longleftarrow$: let $\varphi([\lambda]) := [\lambda P]$; φ is well-defined iff $\big([\lambda] =_{\mathbf{M}} 0 \Rightarrow [\lambda P] =_{\mathbf{M}'} 0\big)$

By exactness: $[\lambda] =_{\mathbf{M}} 0 \quad \Rightarrow \quad \big(\exists \mu \in \mathbf{D}^{1\times q} : \quad \lambda = \mu R\big)$. Hence, we get:

$$[\lambda P] \quad =_{\mathbf{M}'} \quad [\mu RP] \quad =_{\mathbf{M}'} \quad [\mu QR'] \quad =_{\mathbf{M}'} \quad 0$$

Example

Systems: consider the following two systems of PDEs

$$R\xi := \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 \\ 0 & \partial_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} =_{\mathcal{F}} 0 \qquad R'\zeta := \begin{pmatrix} \partial_1 & 0 & 0 \\ \partial_2 & -1 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 1 & \partial_1 \\ 0 & 0 & \partial_2 \\ 0 & \partial_2 & 0 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} =_{\mathcal{F}} 0$$

Morphism: let $\varphi : \mathbf{M} \to \mathbf{M}'$ induced by

$$P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad Q := \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

The following equations hold

$$RP = \begin{pmatrix} \partial_1 & 0 & 0 \\ \frac{1}{2}\partial_2 & 0 & \frac{1}{2}\partial_1 \\ 0 & 0 & \partial_2 \end{pmatrix} = QR'$$

Pullback of solutions:

$$\varphi^\star : \ker_{\mathcal{F}}(R'.) \to \ker_{\mathcal{F}}(R.), \qquad \zeta := \begin{pmatrix} \zeta_1 & \zeta_2 & \zeta_3 \end{pmatrix}^T \mapsto \xi := P\zeta = \begin{pmatrix} \zeta_1 & \zeta_3 \end{pmatrix}^T$$

Theorem [Cluzeau - Quadrat 2011].

Let $\varphi : \mathbf{M} \to \mathbf{M}'$ be a morphism corresponding to matrices P, Q s.t. $RP = QR'$.

Then, φ is an isomorphism iff $\exists\ P' \in \mathbf{D}^{p' \times p}, \quad Q' \in \mathbf{D}^{q' \times q}, \quad Z \in \mathbf{D}^{p \times q}, \quad Z' \in \mathbf{D}^{p' \times q'}$:

$$\begin{array}{ccccccc}
\mathbf{D}^{1\times q} & \xrightarrow{.R} & \mathbf{D}^{1\times p} & \xrightarrow{\pi} & \mathbf{M} & \longrightarrow & 0 \\
\downarrow .Q & & \downarrow .P & & \downarrow \varphi & & \\
\mathbf{D}^{1\times q'} & \xrightarrow[.R']{} & \mathbf{D}^{1\times p'} & \xrightarrow{\pi'} & \mathbf{M}' & \longrightarrow & 0
\end{array}$$

s.t. the following relations hold

$$R'P' = Q'R \qquad PP' + ZR = \mathrm{id}_p \qquad P'P + Z'R' = \mathrm{id}_{p'}$$

Theorem [Cluzeau - Quadrat 2011].

Let $\varphi : \mathbf{M} \to \mathbf{M}'$ be a morphism corresponding to matrices P, Q s.t. $RP = QR'$.

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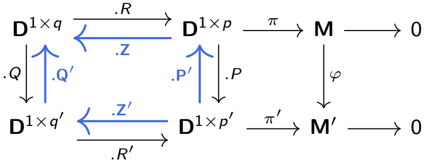

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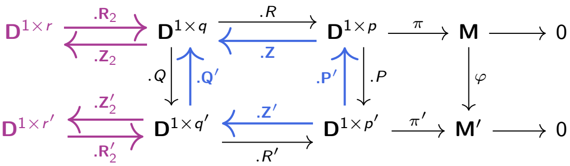


s.t. the following relations hold

$$R'P' = Q'R \qquad PP' + ZR = \mathrm{id}_p \qquad P'P + Z'R' = \mathrm{id}_{p'}$$

Moreover, there exist $R_2 \in \mathbf{D}^{r \times q}, \quad Z_2 \in \mathbf{D}^{q \times r}, \quad R'_2 \in \mathbf{D}^{r' \times q'}, \quad Z'_2 \in \mathbf{D}^{q' \times r'}$ s.t.

$$\ker(.R) = \mathbf{D}^{1 \times r}\, R_2 \qquad QQ' + RZ + Z_2 R_2 = \mathrm{id}_q$$

$$\ker(.R') = \mathbf{D}^{1 \times r}\, R'_2 \qquad Q'Q + R'Z' + Z'_2 R'_2 = \mathrm{id}_{q'}$$

Main steps of the proof (suppose φ is an isomorphism)

Objective: construct an exact diagram and check various relations

$$
\begin{array}{ccccccccc}
\mathbf{D}^{1\times r} & \underset{.Z_2}{\overset{.R_2}{\rightleftarrows}} & \mathbf{D}^{1\times q} & \underset{.Z}{\overset{.R}{\rightleftarrows}} & \mathbf{D}^{1\times p} & \xrightarrow{\pi} & \mathbf{M} & \longrightarrow & 0 \\
 & & {\scriptstyle .Q}\downarrow\uparrow{\scriptstyle .Q'} & & {\scriptstyle .P'}\uparrow\downarrow{\scriptstyle .P} & & \downarrow{\scriptstyle \varphi} & & \\
\mathbf{D}^{1\times r'} & \underset{.R_2'}{\overset{.Z_2'}{\leftrightarrows}} & \mathbf{D}^{1\times q'} & \underset{.R'}{\overset{.Z'}{\leftrightarrows}} & \mathbf{D}^{1\times p'} & \xrightarrow{\pi'} & \mathbf{M}' & \longrightarrow & 0
\end{array}
$$

Main steps of the proof
(suppose φ is an isomorphism)

Objective: construct an exact diagram and check **various relations**

$$
\begin{array}{ccccccccc}
\mathbf{D}^{1\times r} & \underset{.Z_2}{\overset{.R_2}{\rightleftarrows}} & \mathbf{D}^{1\times q} & \underset{.Z}{\overset{.R}{\rightleftarrows}} & \mathbf{D}^{1\times p} & \xrightarrow{\pi} & \mathbf{M} & \longrightarrow & 0 \\
 & & .Q \downarrow \uparrow .Q' & & .P' \uparrow \downarrow .P & & \downarrow \varphi & & \\
\mathbf{D}^{1\times r'} & \underset{.R_2'}{\overset{.Z_2'}{\leftrightarrows}} & \mathbf{D}^{1\times q'} & \underset{.R'}{\overset{.Z'}{\leftrightarrows}} & \mathbf{D}^{1\times p'} & \xrightarrow{\pi'} & \mathbf{M}' & \longrightarrow & 0
\end{array}
$$

$\mathbf{R}'\mathbf{P}' = \mathbf{Q}'\mathbf{R}$: due to the existence of φ^{-}

Main steps of the proof (suppose φ is an isomorphism)

Objective: construct an exact diagram and check **various relations**

$$
\begin{array}{ccccccccc}
\mathbf{D}^{1\times r} & \overset{.R_2}{\underset{.Z_2}{\rightleftarrows}} & \mathbf{D}^{1\times q} & \overset{.R}{\underset{.Z}{\rightleftarrows}} & \mathbf{D}^{1\times p} & \xrightarrow{\pi} & \mathbf{M} & \longrightarrow & 0 \\
 & & .Q \downarrow \uparrow .Q' & & .P' \uparrow \downarrow .P & & \downarrow \varphi & & \\
\mathbf{D}^{1\times r'} & \overset{.Z_2'}{\underset{.R_2'}{\leftrightarrows}} & \mathbf{D}^{1\times q'} & \overset{.Z'}{\underset{.R'}{\leftrightarrows}} & \mathbf{D}^{1\times p'} & \xrightarrow{\pi'} & \mathbf{M}' & \longrightarrow & 0
\end{array}
$$

$R'P' = Q'R$: due to the existence of φ^-

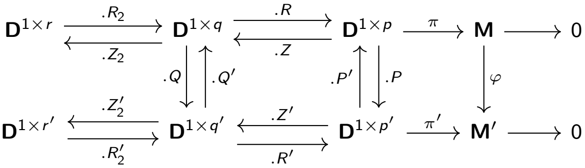
$PP' + ZR = \mathrm{id}_p$: (the proof of $P'P + Z'R' = \mathrm{id}_{p'}$ is analogous)

$$\left(\forall \lambda \in \mathbf{D}^{1\times p}: \; [\lambda] = \varphi^- \circ \varphi([\lambda]) = [\lambda PP']\right) \quad \Rightarrow \quad \mathrm{im}\left(.\left(\mathrm{id}_p - PP'\right)\right) \subseteq \mathbf{D}^{1\times q} R$$

$$\Rightarrow \quad \exists Z \in \mathbf{D}^{p\times q}: \; \mathrm{id}_p - PP' = ZR$$

Main steps of the proof
(suppose φ is an isomorphism)

Objective: construct an exact diagram and check **various relations**



$\mathbf{R'P' = Q'R}$: due to the existence of φ^-

$\mathbf{PP' + ZR = id_p}$: (the proof of $\mathbf{P'P + Z'R' = id_{p'}}$ is analogous)

$$\left(\forall \lambda \in \mathbf{D}^{1\times p}: \; [\lambda] = \varphi^- \circ \varphi([\lambda]) = [\lambda P P']\right) \quad \Rightarrow \quad \operatorname{im}\left(.\left(\operatorname{id}_p - PP'\right)\right) \subseteq \mathbf{D}^{1\times q} R$$

$$\Rightarrow \quad \exists Z \in \mathbf{D}^{p\times q}: \; \operatorname{id}_p - PP' = ZR$$

$\mathbf{QQ' + RZ + Z_2R_2 = id_q}$ and **$\mathbf{Q'Q + R'Z' + Z_2'R_2' = id_{q'}}$:** basic homological algebra

Example

Morphism: $\varphi : \mathbf{M} := \mathbf{D}^{1\times 2} / \mathbf{D}^{1\times 3} R \quad \to \quad \mathbf{M}' := \mathbf{D}^{1\times 3} / \mathbf{D}^{1\times 6} R'$ defined by

$$R := \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 \\ 0 & \partial_2 \end{pmatrix} \qquad R' := \begin{pmatrix} \partial_1 & 0 & 0 \\ \partial_2 & -1 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 1 & \partial_1 \\ 0 & 0 & \partial_2 \\ 0 & \partial_2 & 0 \end{pmatrix} \qquad P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q := \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Fact: φ is an isomorphism, where φ^- is defined by

$$P' := \begin{pmatrix} 1 & 0 \\ \partial_2 & 0 \\ 0 & 1 \end{pmatrix} \qquad Q' := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 2\partial_2 & -\partial_1 \end{pmatrix} \quad \rightsquigarrow \quad R' P' = \begin{pmatrix} \partial_1 & 0 \\ 0 & 0 \\ \partial_1\partial_2 & 0 \\ \partial_2 & \partial_1 \\ 0 & \partial_2 \\ \partial_2^2 & 0 \end{pmatrix} = Q' R$$

Indeed: $\forall \lambda \in \mathbf{D}^{1\times 2} : \quad \varphi^- \circ \varphi([\lambda]) = [\lambda P P'] = [\lambda \, \mathrm{id}_2] = [\lambda]$ and

$$\forall \lambda' \in \mathbf{D}^{1\times 3} : \quad \varphi \circ \varphi^-([\lambda']) = [\lambda' P' P] = \left[\lambda' \begin{pmatrix} 1 & 0 & 0 \\ \partial_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] = [\lambda'_1 + \lambda'_2 \partial_2 \quad 0 \quad \lambda'_3] = [\lambda']$$

III. CONSTRUCTIVE PROOFS OF FITTING AND WARFIELD'S THEOREMS

III.1. FITTING'S THEOREM

Preliminaries

Fix $\big(\mathbf{M} := \mathbf{D}^{1\times p} \,/\, \mathbf{D}^{1\times q}\, R \quad \mathbf{M} := \mathbf{D}^{1\times p'} \,/\, \mathbf{D}^{1\times q'}\, R'\big) \quad \leftrightarrow \quad \big(R\eta =_{\mathcal{F}} 0 \quad R'\eta' =_{\mathcal{F}} 0\big)$

Lemma: $\mathbf{M} \simeq \mathbf{D}^{1\times c} \,/\, \mathbf{D}^{1\times r}\, L$ and $\mathbf{M}' \simeq \mathbf{D}^{1\times c} \,/\, \mathbf{D}^{1\times r}\, L'$, where

$$\begin{cases} c := p + p' \\ r := q + p' + p + q' \end{cases} \qquad L := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \end{pmatrix} \leftarrow (p+q') \qquad L' := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix} \leftarrow (q+p')$$

Extended systems: L and L' correspond to the systems

$$L \begin{pmatrix} \eta \\ \xi \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} R\eta \\ \xi \\ 0 \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad\qquad L' \begin{pmatrix} \eta' \\ \xi' \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} R'\eta' \\ \xi' \\ 0 \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Preliminaries

Fix $\big(\mathbf{M} := \mathbf{D}^{1\times p} / \mathbf{D}^{1\times q} R \quad \mathbf{M} := \mathbf{D}^{1\times p'} / \mathbf{D}^{1\times q'} R'\big) \quad \leftrightarrow \quad \big(R\eta =_{\mathcal{F}} 0 \quad R'\eta' =_{\mathcal{F}} 0\big)$

Lemma: $\mathbf{M} \simeq \mathbf{D}^{1\times c} / \mathbf{D}^{1\times r} L$ and $\mathbf{M}' \simeq \mathbf{D}^{1\times c} / \mathbf{D}^{1\times r} L'$, where

$$\begin{cases} c := p + p' \\ r := q + p' + p + q' \end{cases} \qquad L := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \end{pmatrix} \leftarrow (p+q') \qquad L' := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix} \leftarrow (q+p')$$

Extended systems: L and L' correspond to the systems

$$L \begin{pmatrix} \eta \\ \xi \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} R\eta \\ \xi \\ 0 \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad L' \begin{pmatrix} \eta' \\ \xi' \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} R'\eta' \\ \xi' \\ 0 \end{pmatrix} =_{\mathcal{F}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Fitting's theorem: if $\mathbf{M} \simeq \mathbf{M}'$, then L and L' are equivalent

Example

Isomorphic modules: correspond to the matrices

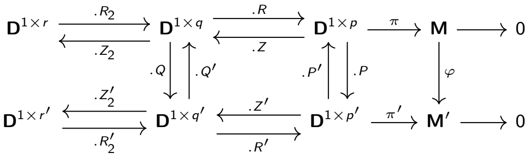
$$R := \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 \\ 0 & \partial_2 \end{pmatrix} \qquad R' := \begin{pmatrix} \partial_1 & 0 & 0 \\ \partial_2 & -1 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 1 & \partial_1 \\ 0 & 0 & \partial_2 \\ 0 & \partial_2 & 0 \end{pmatrix}$$

From Fitting's theorem: $L, L' \in \mathbf{D}^{5\times 14}$ are equivalent matrices

$$L := \begin{pmatrix} \partial_1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 & 0 & 0 & 0 \\ 0 & \partial_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad L' := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \partial_1 & 0 & 0 \\ 0 & 0 & \partial_2 & -1 & 0 \\ 0 & 0 & 0 & \partial_1 & 0 \\ 0 & 0 & 0 & 1 & \partial_1 \\ 0 & 0 & 0 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_2 & 0 \end{pmatrix}$$

Theorem [Cluzeau - Quadrat 2011].

Let $\varphi : \mathbf{M} := \mathbf{D}^{1\times p} / \mathbf{D}^{1\times q} R \xrightarrow{\sim} \mathbf{M}' := \mathbf{D}^{1\times p'} / \mathbf{D}^{1\times q'} R'$ corresponding to



Then, the pairs of invertible matrices $(X, X^-) \in \mathbf{D}^{c\times c}$ and $(Y, Y^-) \in \mathbf{D}^{r\times r}$

$$X := \begin{pmatrix} \mathrm{id}_p & P \\ -P' & \mathrm{id}_{p'} - P'P \end{pmatrix} \qquad X^- := \begin{pmatrix} \mathrm{id}_p - PP' & -P \\ P' & \mathrm{id}_{p'} \end{pmatrix}$$

$$Y := \begin{pmatrix} \mathrm{id}_q & 0 & R & Q \\ 0 & \mathrm{id}_{p'} & -P' & Z' \\ -Z & P & 0 & PZ' - ZQ \\ -Q' & -R' & 0 & Z'_2 R'_2 \end{pmatrix} \qquad Y^- := \begin{pmatrix} Z_2 R_2 & 0 & -R & -Q \\ P'Z - Z'Q' & 0 & P' & -Z' \\ Z & -P & \mathrm{id}_p & 0 \\ Q' & R' & 0 & \mathrm{id}_{q'} \end{pmatrix}$$

are such that

$$LX := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \end{pmatrix} X \quad = \quad Y \begin{pmatrix} 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix} =: YL'$$

Comments

Ideas of the proof: compute matrix products using relations between P's, Q's, R's, Z's

Improvement: if R and R' have full row rank, we may remove the 0 lines of L and L'

Practical use:

- ➔ start with the two modules presented by R and R'
- ➔ if an isomorphism exists, compute P, P', Q, and Q'
- ➔ compute Z, Z', R_2, R_2', Z_2, and Z_2'
- ➔ deduce the expressions of X, X^-, Y, and Y^-

Implementation: package `OreMorphisms` (*Maple*) when $\mathbf{D}$ is an Ore algebra

- ➔ ODE/PDEs operators, shift operators, time-delay operators, difference operators

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- ➔ ODE/PDEs operators, shift operators, time-delay operators, difference operators

Running example

$$X = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ -\partial_2 & 0 & -\partial_2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \qquad X^{-} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 \\ \partial_2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \partial_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2}\,\partial_2 & \frac{1}{2}\,\partial_1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\partial_2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\partial_2 & 0 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -1 & -\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\,\partial_2 & \partial_1 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 & \partial_1 & 1 \end{pmatrix}$$

Running example (end)

$$
Y^{-} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -\partial_1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\,\partial_2 & -\frac{1}{2}\,\partial_1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & \partial_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & \partial_2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\partial_2 & -2\,\partial_1 & 0 & 0 & \partial_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \partial_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -\partial_1 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Running example (end)

$$Y^- = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -\partial_1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\,\partial_2 & -\frac{1}{2}\,\partial_1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & \partial_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & \partial_2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\partial_2 & -2\,\partial_1 & 0 & 0 & \partial_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \partial_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -\partial_1 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Summary: $X, X^- \in \mathbf{D}^{5\times 5}$ and $Y, Y^- \in \mathbf{D}^{14\times 14}$

III.2. WARFIELD'S THEOREM

Next objective

if $\mathbf{M} \simeq \mathbf{M}'$: reduce the size of L and L'

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Warfield's theorem: Let $i, z \in \mathbb{N}$ be such that

$$\begin{cases} z \leq \min(p + q', q + p') \\ \mathrm{sr}(\mathbf{D}) \leq \max(p + q' - z, q + p' - z) \\ i \leq \min(p, p') \\ \mathrm{sr}(\mathbf{D}) \leq \max(p - i, p' - i) \end{cases}$$

where $\mathrm{sr}(\mathbf{D})$ is the stable rank of $\mathbf{D}$. If $\mathbf{M} \simeq \mathbf{M}'$, then the following matrices are equivalent

$$\tilde{L} := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'-i} \\ 0 & 0 \end{pmatrix} \leftarrow (p + q' - z) \qquad \tilde{L}' := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{p-i} & 0 \\ 0 & R' \end{pmatrix} \leftarrow (q + p' - z)$$

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where $\mathrm{sr}(\mathbf{D})$ is the **stable rank** of $\mathbf{D}$. If $\mathbf{M} \simeq \mathbf{M}'$, then the following matrices are equivalent

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Stable rank

Preliminaries: let $u := \begin{pmatrix} u_1 & \dots & u_n \end{pmatrix}^T \in \mathbf{D}^n$ be a column vector

- u is called unimodular if there exists a row vector $v \in \mathbf{D}^{1 \times n}$ s.t. $vu = 1$
- u is called stable if there exists $d_1, \dots, d_{n-1} \in \mathbf{D}$ s.t.

$$\begin{pmatrix} u_1 + d_1\, u_n & \dots & u_{n-1} + d_{n-1} u_n \end{pmatrix}^T \text{ is unimodular}$$

Definition: the stable rank of $\mathbf{D}$ is the smallest integer $\mathrm{sr}(\mathbf{D})$ s.t.

$$\forall n > \mathrm{sr}(\mathbf{D}) : \quad u \in \mathbf{D}^n \text{ is unimodular} \quad \Rightarrow \quad u \text{ is stable}$$

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Examples:

- If $\mathbf{D}$ is a principal domain, then $\mathrm{sr}(\mathbf{D}) \geq 2$
- For every $n \geq 1$, $\mathrm{sr}(\mathbb{Q}[x_1, \dots, x_n]) = n + 1$
- Stafford's theorem: if $\mathrm{char}(\mathbb{K}) = 0$, then $\mathrm{sr}(A_n(\mathbb{K})) = \mathrm{sr}(B_n(\mathbb{K})) = 2$
  $(A_n(\mathbb{K})/B_n(\mathbb{K}) :=$ polynomial/rational Weyl algebra$)$

Removing one zero line

From Fitting's theorem: there is a commutative diagram

$$\begin{array}{ccc} \mathbf{D}^{1\times r} & \xrightarrow{.Y} & \mathbf{D}^{1\times r} \\ {\scriptstyle .L}\downarrow & & \downarrow{\scriptstyle .L'} \\ \mathbf{D}^{1\times c} & \xrightarrow[.X]{} & \mathbf{D}^{1\times c} \end{array}$$

Objective: prove that L_1 and L_1' are equivalent

$$L_1 := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \end{pmatrix} \leftarrow \ (p + q' - 1) \qquad L_1' := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix} \leftarrow \ (q + p' - 1)$$

Method: construct a commutative diagram s.t. the top row is invertible

$$\begin{array}{ccccccccc} \mathbf{D}^{1\times(r-1)} & \xrightarrow{.H_1} & \mathbf{D}^{1\times r} & \xrightarrow{.W_1} & \mathbf{D}^{1\times r} & \xrightarrow{.Y} & \mathbf{D}^{1\times r} & \xrightarrow{.G_1} & \mathbf{D}^{1\times(r-1)} \\ {\scriptstyle .L_1}\downarrow & & \downarrow{\scriptstyle .L} & & \downarrow{\scriptstyle .L} & & \downarrow{\scriptstyle L'} & & \downarrow{\scriptstyle .L_1'} \\ \mathbf{D}^{1\times c} & = & \mathbf{D}^{1\times c} & = & \mathbf{D}^{1\times c} & \xrightarrow[.X]{} & \mathbf{D}^{1\times c} & = & \mathbf{D}^{1\times c} \end{array}$$

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$$L_1 := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \end{pmatrix} \leftarrow (p+q'-1) \qquad L_1' := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix} \leftarrow (q+p'-1)$$

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➔ "extract" an invertible codimension 1 matrix from Y

Lemma: if there exist $\mathbf{c} \in \mathbf{D}$, $\mathbf{v} \in \mathbf{D}^{1\times(p+q'-1)}$, and $\mathbf{u} \in \mathbf{D}^{p+q'-1}$ s.t.

$$\mathbf{c} \sum_{j=1}^{q+p'} Y^{-}_{(q+p')j}\, Y_{j(q+p')} + \sum_{j=q+p'+1}^{r-1} \mathbf{v}_{\bar{j}} \left(Y_{j(q+p')} + \mathbf{u}_{\bar{j}} Y_{r(q+p')} \right) = 1 \qquad \left(\bar{j} := j-(q+p')\right)$$

then, there exists a commutative diagram s.t. the top row is invertible

$$\begin{array}{ccccccccc}
\mathbf{D}^{1\times(r-1)} & \xrightarrow{.H_1} & \mathbf{D}^{1\times r} & \xrightarrow{.W_1} & \mathbf{D}^{1\times r} & \xrightarrow{.Y} & \mathbf{D}^{1\times r} & \xrightarrow{.G_1} & \mathbf{D}^{1\times(r-1)} \\
\downarrow {\scriptstyle .L_1} & & \downarrow {\scriptstyle .L} & & \downarrow {\scriptstyle .L} & & \downarrow {\scriptstyle L'} & & \downarrow {\scriptstyle .L_1'} \\
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\end{array}$$

Moreover, $\mathbf{c}$, $\mathbf{v}$, and $\mathbf{u}$ exist as soon as

$$\begin{cases} 1 \leq \min(p+q', q+p') \\ \mathrm{sr}(\mathbf{D}) \leq \max(p+q'-1, q+p'-1) \end{cases}$$

Ideas of the proof

Notations: consider $\ell \in \mathbf{D}^{1\times(r-1)}$ and $F \in \mathbf{D}^{(r-1)\times r}$

$$\ell := \begin{pmatrix} \mathbf{c}(Y_1')_{(q+p').} & \mathbf{v} \end{pmatrix} \qquad F := \begin{pmatrix} Y_1 \\ Y_2 + \mathbf{u}\, Y_3 \end{pmatrix}$$

where Y_1, Y_2, Y_3 (resp., Y_1', Y_2', Y_3') have $q+p'$, $p+q'-1$, and 1 lines (resp., columns) and are s.t.

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \qquad Y^- = \begin{pmatrix} Y_1' & Y_2' & Y_3' \end{pmatrix}$$

Construction of the diagram: we define the matrices

$$G_1 := \begin{pmatrix} \mathrm{id}_{r-1} - F_{.(q+p')}\ell \\ \ell \end{pmatrix} \quad H_1 := \begin{pmatrix} \mathrm{id}_{r-1} - F_{.(q+p')}\ell & F_{.(q+p')} \end{pmatrix} \quad W_1 := \begin{pmatrix} \mathrm{id}_{q+p'} & 0 & 0 \\ 0 & \mathrm{id}_{p+q'-1} & \mathbf{u} \\ 0 & 0 & 1 \end{pmatrix}$$

Commutativity/invertibility: matrix computations using the relation satisfied by **c**, **v**, and **u**

Stable rank condition: lemma on quotients of free modules (NOT ALGORITHMIC!!!)

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Commutativity/invertibility: matrix computations using the relation satisfied by **c**, **v**, and **u**

Stable rank condition: lemma on quotients of free modules (**NOT ALGORITHMIC!!!**)

Removing many 0 lines

Notation: for $k \leq \min(q+p', p+q')$, let L_k and L'_k be defined by

$$L_k := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \end{pmatrix} \leftarrow (p+q'-k) \qquad L'_k := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix} \leftarrow (q+p'-k)$$

Induction: assume that an invertible matrix $Y^{(k-1)}$ has been constructed s.t. $L_{k-1}X = Y^{(k-1)}L'_{k-1}$

Lemma: if $\exists \mathbf{c} \in \mathbf{D}$, $\mathbf{v} \in \mathbf{D}^{1\times(p+q'-k+1)}$, and $\mathbf{u} \in \mathbf{D}^{p+q'-k+1}$ s.t.

$$\mathbf{c} \sum_{j=1}^{q+p'} \left(Y^{(k-1)}\right)^{-}_{(q+p'-k+1)j} Y^{(k-1)}_{j(q+p'-k+1)} + \sum_{j=q+p'+1}^{r-k} \mathbf{v}_{\bar{j}} \left(Y^{(k-1)}_{j(q+p'-k+1)} + \mathbf{u}_{\bar{j}} Y^{(k-1)}_{(r-k+1)(q+p'-k+1)}\right) = 1$$

then, we can construct $Y^{(k)} := H_k W_k Y^{(k-1)} G_k$ invertible s.t. $L_k X = Y^{(k)} L'_k$. Moreover, $\mathbf{c}$, $\mathbf{v}$, and $\mathbf{u}$ exist as soon as

$$\begin{cases} k \leq \min(p+q', q+p') \\ \mathrm{sr}(\mathbf{D}) \leq \max(p+q'-k, q+p'-k) \end{cases}$$

Theorem [C., Cluzeau, Quadrat].

Let $\varphi : \mathbf{M} := \mathbf{D}^{1\times p} / \mathbf{D}^{1\times q} R \quad \to \quad \mathbf{M}' := \mathbf{D}^{1\times p} / \mathbf{D}^{1\times q} R$ be an iso. and let $z \in \mathbb{N}$ s.t.

$$z \leq \min(p+q', q+p') \qquad \text{and} \qquad \operatorname{sr}(\mathbf{D}) \leq \max(p+q'-z, q+p'-z)$$

Then, the following matrices are equivalent

$$L_z := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \end{pmatrix} \leftarrow (p+q'-z) \qquad L'_z := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix} \leftarrow (q+p'-z)$$

and there is an inductive procedure that constructs an invertible matrix $Y^{(z)}$ s.t. $L_z X = Y^{(z)} L'_z$.

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Let $\varphi : \mathbf{M} := \mathbf{D}^{1\times p} \,/\, \mathbf{D}^{1\times q}\, R \quad \to \quad \mathbf{M}' := \mathbf{D}^{1\times p} \,/\, \mathbf{D}^{1\times q}\, R$ be an iso. and let $z \in \mathbb{N}$ s.t.

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Similarly if $i \in \mathbb{N}$ is s.t.

$$i \leq \min(p, p') \qquad \text{and} \qquad \mathrm{sr}(\mathbf{D}) \leq \max(p-i, p'-i)$$

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$$\tilde{L} := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'-i} \\ 0 & 0 \end{pmatrix} \leftarrow (p+q'-z) \qquad\qquad \tilde{L}' := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{p-i} & 0 \\ 0 & R' \end{pmatrix} \leftarrow (q+p'-z)$$

and there is an inductive procedure that constructs invertible matrices $\tilde{X}, \tilde{Y}$ s.t. $\tilde{L}\tilde{X} = \tilde{Y}\tilde{L}'$.

Example

Continue the previous example

$$R := \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 \\ 0 & \partial_2 \end{pmatrix} \qquad R' := \begin{pmatrix} \partial_1 & 0 & 0 \\ \partial_2 & -1 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 1 & \partial_1 \\ 0 & 0 & \partial_2 \\ 0 & \partial_2 & 0 \end{pmatrix}$$

$$L := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_3 \\ 0 & 0 \end{pmatrix} \leftarrow 8 \qquad L' := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_2 & 0 \\ 0 & R' \end{pmatrix} \leftarrow 6$$

Theoretically: we can remove 5 lines of 0 and 0 identity block since

- ➔ $\mathrm{sr}(\mathbf{D}) = \mathrm{sr}(\mathbb{Q}[\partial_1, \partial_2]) = 3$
- ➔ $z \leq \min(p + q', q + p') = 6$ and $\mathrm{sr}(\mathbf{D}) \leq \max(p + q' - z, p + q' - z)$
- ➔ $i \leq \min(p, p') = 2$ and $\mathrm{sr}(\mathbf{D}) \leq \max(p - i, p' - i)$

Practically: using obvious solutions and heuristics to compute **c**, **v**, and **u**'s

- ➔ we remove 5 lines of 0 and 2 identity blocks

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Continue the previous example

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Example (end)

Summary: we obtained $\tilde{L}\tilde{X} = \tilde{Y}\tilde{L}'$ where

$$\tilde{L} := \begin{pmatrix} \partial_1 & \mathbf{0} & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 & 0 \\ \mathbf{0} & \partial_2 & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \tilde{L}' := \begin{pmatrix} 0 & 0 & 0 \\ \partial_1 & \mathbf{0} & \mathbf{0} \\ \partial_2 & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \partial_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \partial_1 \\ \mathbf{0} & \mathbf{0} & \partial_2 \\ \mathbf{0} & \partial_2 & \mathbf{0} \end{pmatrix}$$

and the invertible matrices

$$\tilde{X} = \begin{pmatrix} 1-\partial_2 & 1 & 0 \\ 0 & 0 & 1 \\ \partial_2 & -1 & 0 \end{pmatrix} \qquad \tilde{X}^- = \begin{pmatrix} 1 & 0 & 1 \\ \partial_2 & 0 & -1+\partial_2 \\ 0 & 1 & 0 \end{pmatrix} \qquad \tilde{Y}, \tilde{Y}^-$$

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Open question: is it possible to remove the last 0 line?

IV. CONCLUSION AND PERSPECTIVES

Conclusion and perspectives

We presented: a constructive approach to module theory

- ▷ matrix descriptions of module (iso)morphisms
- ▷ a constructive proof of Fitting's theorem
- ▷ a procedure that make the Warfield theorem almost constructive

Further work: make the proof of Warfield's theorem completely constructive

- ▷ at least for sufficiently nice rings/systems (ODE, (q)-differences, PDE)
- ▷ remove the last 0 line in the running example

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THANK YOU FOR LISTENING!