Normal forms of matrix words for stability analysis of discrete-time switched linear systems

Cyrille Chenavier[1] Rosane Ushirobira[2] Laurentiu Hetel[3]

[1]Johannes Kepler Universität, Linz, Austria [2]Inria, France [3]CNRS, U. Lille, France

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Discrete-time switched linear systems

A discrete-time switched linear system is given by

$$x_{k+1} = A_{\sigma(k)} x_k, \qquad k \in \mathbb{N}, \quad x_0 \in \mathbb{R}^n$$

where

- $x : \mathbb{N} \to \mathbb{R}^n$ represents the state variable, $x(0) = x_0$ is the initial state
- $A_1, \cdots, A_p \in \mathbb{R}^{n \times n}$ are matrices representing stable subsystems
- $\sigma : \mathbb{N} \to \{A_1, \cdots, A_p\}$ is the switching function (not known)

Problem

Analyse global uniform exponential stability (GUES) of such systems

$\rightsquigarrow$ do any trajectory converges to 0 with exponential decay?

Existing stability analysis methods

- Joint spectral radius (Blondel)
- Lie algebraic conditions (Liberzon, Gurvitz)
- **Set theoretic approach** (Megretski, Kruszewski, Guerra)
- Lyapunov functions (sufficient condition)

Megretski's method

- Requires to solve LMIs problem
- LMIs are indexed by **matrix words**

Trajectories

The trajectory associated to the switching

$$\sigma(1) = i_1 \in \{1, \cdots, p\}, \quad \sigma(2) = i_2 \in \{1, \cdots, p\}, \cdots$$

has the form

$$x_0 \to A_{i_1} x_0 \to A_{i_2} A_{i_1} x_0 \to A_{i_3} A_{i_2} A_{i_1} x_0 \to \cdots$$

Matrix representation of finite trajectories

If $w = i_k \cdots i_1$ is a k-length word over $\{1, \cdots, p\} \rightsquigarrow A_w := A_{i_k} \cdots A_{i_2} A_{i_1}$

- Example:

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  then

$$A_{11} = A_1 A_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad A_{12} = A_1 A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{21} = A_2 A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

$$A_{22} = A_2 A_2 = \mathrm{Id}_2$$

Theorem [Megretski, '97]

The discrete-time switched linear system is GUES if and only if

$\exists N > 0$ and $P = P^T \succ 0$ s.t. the following LMIs problem admits a solution

$$P \succ A_w^T P A_w, \qquad \forall w = i_N \cdots i_1$$

Remark: the size of LMIs grows exponentially (p^N words of length N)

Contribution of the work

Use linear algebra methods to reduce the size of the LMIs problem

- Motivation: assume that $P = P^T \succ 0$ solves LMIs for $A_{w_1}, \cdots A_{w_r}$ and

$$A_{w_0} = \sum_{i=1}^{r} \lambda_i A_{w_i}$$

Under which conditions P also solves the LMI for A_{w_0}?

Definition

Let N be an integer and

$$d_N := \dim\left(Vect(A_w : \ w = i_N \cdots i_1)\right) \quad \subseteq \quad \mathbb{R}^{n\times n}$$

A free set of matrices $A_{w_1}, \cdots A_{w_{d_N}}$ is called a set of **normal form matrices**

Remark

If A_w is not a normal form matrix, it admits a unique decomposition

$$A_w = \sum_{i=1}^{d_N} \lambda_i^w A_{w_i}$$

Question: how to use linear algebra to restrict LMIs to normal form matrices?

1st candidate for a new LMIs problem

Let $N > 0$ and $A_{w_1}, \cdots A_{w_{d_N}}$ be normal form matrices

$$\exists P = P^T \succ 0 \text{ s.t. } P \succ A_{w_i}^T P A_{w_i}, \qquad 1 \leq i \leq d_N$$

Remark: the number of LMIs is bounded by the constant n^2 $(d_N \leq n^2)$

Problem

If the decomposition of a non normal form matrix

$$A_w = \sum_{i=1}^{d_N} \lambda_i^w A_{w_i}$$

involves "big" coefficients, $P \succ A_w^T P A_w$ does not hold

The LMIs problem has to take λ_i^w's into account

Lemma

If P is a solution to the LMIs problem

$$\exists P = P^T \succ 0 \text{ s.t. } P \succ A_i^T P A_i, \qquad 1 \leq i \leq d_N$$

Then, $P \succ A^T P A$ holds for every A is the convex hull of A_i's

2nd candidate for a new LMIs problem

Let $N > 0$ and $A_{w_1}, \cdots, A_{w_{d_N}}$ be normal form matrices

$$\exists P = P^T \succ 0 \text{ s.t. } P \succ \mu_i A_{w_i}^T P A_{w_i}, \qquad 1 \leq i \leq d_N$$

where μ_i's are such that

"linear combinations are transformed into convex decompositions"

From linear to convex decompositions

Start with a linear combination of a non normal form matrix

$$A_w = \sum_{i=1}^{d_N} \lambda_i^w A_{w_i}$$

Letting $n_w := \mid \lambda_1^w \mid + \cdots + \mid \lambda_n^w \mid$, we get the following convex decomposition

$$A_w = \sum_{i=1}^{d_N} \frac{\mid \lambda_i^w \mid}{n_w} (\varepsilon(\lambda_i^w) n_w A_{w_i})$$

Choices for μ_i's

First choice: all μ_i's are equal to $\max(n_w : A_w$ is not a normal form matrix$)$

A more optimal choice: $\mu_i = \max(n_w : A_w$ is not a normal form matrix and $\lambda_i^w \neq 0)$

Theorem

Consider the discrete-time switched linear system

$$x_{k+1} = A_{\sigma(k)} x_k, \qquad k \in \mathbb{N}, \quad x_0 \in \mathbb{R}^n \tag{1}$$

Let N be a strictly positive integer and let $A_1, \cdots, A_{d_N}$ be normal form matrices. For every non normal form matrix A_w, let us consider its unique decomposition

$$A_w = \sum_{i=1}^{d_N} \lambda_i^w A_{w_i}$$

and for every $1 \leq i \leq d_N$, let

$$\mu_i := \max(n_w : A_w \text{ is not a normal form matrix and } \lambda_i^w \neq 0)$$

If the following LMIs problem admits a solution

$$\exists P = P^T \succ 0 \text{ s.t. } P \succ \mu_i A_{w_i}^T P A_{w_i}, \qquad 1 \leq i \leq d_N$$

then (1) is GUES

Example

Consider the discrete-time switched linear system defined with $p = 2$ and $A_i = \exp(A_i^c T)$, with $T = 1$, where

$$A_1^c = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \qquad A_2^c = \begin{pmatrix} -1 & -a \\ \frac{1}{a} & -1 \end{pmatrix}$$

Changing the value of the parameter a, we get

	a=5	a=6	a=7	a=8	#LMI conditions
N=1	✓	-	-	-	2
N=3	✓	✓	-	-	9
N=8	✓	✓	✓	✓	257

where ✓ means that a solution to the LMIs problem was obtained, and $-$ not

- We investigated stability of discrete-time switched linear systems using linear algebra techniques
- Our approach may be used to reduce drastically the number of LMI's conditions to check stability
- The counter-part of the approach is that LMI's are have higher numerical constraints

THANK YOU!