Introduction to linear rewriting:

Gröbner bases and reduction operators

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Algebraic rewriting Seminar

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Université de Limoges

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I. Motivations

- ▷ Effective algebraic computation
- ▷ Formalisation of algebraic computation

II. Commutative Gröbner bases

- ▷ Polynomial reduction and Gröbner bases
- ▷ Completion algorithms

III. Noncommutative Gröbner bases

- ▷ Noncommutative polynomial reduction
- ▷ Anick's resolution and Koszulness

IV. Reduction operators

- ▷ Functional representation of linear rewriting systems
- ▷ Lattice characterisation of confluence and completion

I. MOTIVATIONS

Effective algebraic computation

Objective: compute with (non)commutative/Lie/tree polynomials

➔ membership problem, computation of representatives and linear bases

Application scopes: algebraic geometry/combinatorics, homological algebra, formal analysis of functional equations, cryptography

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ALGEBRAIC REWRITING

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Formalisation of algebraic computation

Paradigms of rewriting: Gröbner bases and adaptations, linear polygraphs, reduction operators

Algebraic tools for rewriting: monomial orders, critical pairs, higher-dimensional rewriting strategies

Some algorithmic problems in algebra

- solve decision problems (*e.g.*, membership problem)
- compute homological invariants (*e.g.*, Tor, Ext groups)
- analysis of functional systems (*e.g.*, integrability conditions)

Classical ⟵ **techniques**

Constructive methods in algebra

- compute set of representatives for congruence classes
- construct free resolutions of modules
- elimination theory for systems of equations

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Approach: orientation of relations ➔ notion of normal form

example: chosen orientation in $\mathbb{K}[x, y]$ ➔ induced by $yx \to xy$

NF computation: $3\ yxx + xyx - xy \to 4\ xyx - xy \to 4\ xxy - xy$

Remark on the case $\mathbb{K}[x, y]$ **:** NF monomials $x^n y^m$ form a linear basis

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MOTIVATING PROBLEM

Given an algebra $\mathbf{A} := \mathbb{K}\langle X \mid R\rangle$ presented by generators X and relations R

$$\mathbf{A} := \mathbb{K}\langle X\rangle / I(R) \qquad \Big(e.g., \quad \mathbb{K}[x,y] = \mathbb{K}\langle x, y \mid yx - xy\rangle \Big)$$

Question: given an orientation of R (*e.g.*, $yx \to xy$)

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Equivalently

do NF monomials form a generating family?

do NF monomials form a free family?

NF monomials do not form a generating family

$\mathbf{A} := \mathbb{K}\langle x \mid x - xx\rangle$ orientation: $x \to xx$

➔ $\dim_{\mathbb{K}}(\mathbf{A}) = 2$ $\left(\overline{1} \text{ and } \overline{x} \text{ form a basis}\right)$

➔ 1 is the only NF monomial $\left(\forall n \geq 1 : \quad x^n \to x^{n+1}\right)$

Definition: $\to$ is called terminating if

$\nexists$ infinite rewriting sequence

$$f_1 \to f_2 \to \cdots \to f_n \to f_{n+1} \to \dots$$

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NF monomials are generators

Prop: let $\mathbf{A} := \mathbb{K}\langle X \mid R \rangle$. If $\to$ is a terminating orientation, then

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"termination $\leftrightarrow$ generating"

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NF monomials are not free

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\begin{tikzcd}
 & yyy \arrow[dl] \arrow[dr] & \\
yxy & & yyx \arrow[dl] \\
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\end{tikzcd}
\Longrightarrow
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\overline{yxy} = \overline{yxx} \\
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yxy,\ yxx \text{ are } \neq \\
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Definition: $\to$ is called confluent if

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"confluence $\leftrightarrow$ freeness"

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Monomial orders

Well-founded total orders on X^*, product compatible

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Induces for A $:= \mathbb{K}\langle X \mid R\rangle$

Natural orientation

$\forall f = \mathrm{lc}(f)\,\mathrm{lm}(f) - \mathrm{rem}(f) \in R$

$\mathrm{lm}(f) \to_R 1/\,\mathrm{lc}(f)\,\mathrm{rem}(f)$

Gröbner bases definition

R is called a G.B. of $I = I(R)$ if

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Relationship

Gröbner bases definition

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Theorem. Let I be a (noncommutative) polynomial ideal, R be a generating set of I, and $<$ be a monomial order. Then,

R is a Gröbner basis of $I \quad \Leftrightarrow \quad \to_R$ is a confluent orientation

Objective and plan of the talk

Sections II and III: basics of (noncommutative) Gröbner bases

- ➔ define Gröbner bases in terms of monomials ideals
- ➔ show rewriting characterisation of Gröbner bases
- ➔ present completion algorithms and Anick's resolution

Remark. Gröbner bases have adaptations to many other structures, *e.g.*, Lie algebras, operads, Weyl/Ore algebras, tensor rings

Section IV: introduction to reduction operators

- ➔ definition of reduction operators for vector spaces
- ➔ lattice characterisations of confluence and completion

II. COMMUTATIVE GRÖBNER BASES

Question: given $I \subseteq \mathbb{K}[X]$ and $g \subseteq \mathbb{K}[X]$

how to compute $f \mod I$?

USING REWRITING!

Case of one variable: we recover Euclidean division, *e.g.*,

$$f := x^4 + 3x^3 + 2x + 1 \qquad \text{and} \qquad g := x^2 + 1$$

$f \mod (g)$ is computed by reducing f into NF w.r.t. $x^2 \to -1$

$$x^4 + 3x^3 + 2x + 1 \longrightarrow 3x^3 - x^2 + 2x + 1 \longrightarrow -x^2 - x + 1 \longrightarrow -x + 2$$

➔ $f = \left(x^2 + 3x - 1\right).\left(x^2 + 1\right) - x + 2$

Case of many variables: requires a suitable notion of "leading monomial"

➔ based on monomial orders

Definition: a monomial order (on the set of commutative monomials $[X]$) is an order $<$ on $[X]$ s.t.

$<$ is **total**, **well-founded** and **admissible**, *i.e.*,

$$\forall m, m_1, m_2 \in [X] : \quad m_1 < m_2 \;\Rightarrow\; mm_1 < mm_2$$

INDUCED NOTIONS

Leading monomial, leading coefficient and remainder

$\forall f \in \mathbb{K}[X] \setminus \{0\}$, we define

- ➔ the leading monomial $\mathrm{lm}(f)$ of f as being $\max(\mathrm{supp}(f))$
- ➔ the leading coefficient $\mathrm{lc}(f)$ of f as being the coefficient of $\mathrm{lm}(f)$ in f
- ➔ the remainder of f by $\mathrm{rem}(f) = \mathrm{lc}(f)\,\mathrm{lm}(f) - f$

Generalisation of Euclidean division

Let $f, g \in \mathbb{K}[X]$ and $G \subseteq \mathbb{K}[X]$

Reducing f w.r.t. g: if $f = \lambda m + f'$, with $m = \mathrm{lm}(g)m'$, $m \notin \mathrm{supp}(f')$

and $\lambda \neq 0$, then, we have: $f \to_g \frac{\lambda}{\mathrm{lc}(f)}\Big(m'.\,\mathrm{rem}(g)\Big) + f'$

Reducing f w.r.t. G: $f \to_G f'$ iff $\exists g \in G : \quad f \to_g f'$

A NF of f for $\to_G$ is also called a remainder of f w.r.t. G

THE REMAINDER IS NOT UNIQUE IN GENERAL

Example: $G := \{g_1, g_2\}$ with $g_1 := xy^2 + x$ and $g_2 := 2y^3 + xy - 1$

➔ xy^3 has two remainders: $-xy$ and $-x/2.\big(xy - 1\big)$

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Definition: let $I \subseteq \mathbb{K}[X]$ and let $<$ be a monomial order on $[X]$.

A (commutative) **Gröbner basis of I** is a subset $G \subseteq I$ s.t.

G is a generating set of I and $\mathbf{lm(I)} = \langle \mathbf{lm(G)} \rangle$

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Proposition: the following assertions are equivalent

- G is a Gröbner basis of I
- $\forall f \in I, \quad \exists g \in G : \quad \mathrm{lm}(g) \mid \mathrm{lm}(f)$
- there is a vector space isomorphism $\mathbb{K}[X]/I \simeq \mathbb{K}\,\mathrm{NF}(G)$
- every $f \in \mathbb{K}[X]$ admits a unique remainder w.r.t. G
- every S-polynomial of G rewrites into 0

Definition of S-polynomials

The S-polynomial of $g, g' \in G$ is

$$\frac{\mathrm{lcm}(\mathrm{lm}(g), \mathrm{lm}(g'))}{\mathrm{lm}(g)} g - \frac{\mathrm{lcm}(\mathrm{lm}(g), \mathrm{lm}(g'))}{\mathrm{lm}(g')} g'$$

Theorem. Let I be a polynomial ideal, G be a generating set of I, and $<$ be a monomial order. Then,

$$G \text{ is a Gröbner basis of } I \quad \Leftrightarrow \quad \text{the polynomial reduction is confluent}$$

Ideas of the proof

Step 1: G is a G.B. of I $\quad\Leftrightarrow\quad \mathrm{lm}(I) \cap \mathrm{NF}(G) = \emptyset \quad\Leftrightarrow\quad \mathbb{K}[X] = I \oplus \mathbb{K}\,\mathrm{NF}(G)$

Step 2: $\to_G$ is confluent $\quad\Leftrightarrow\quad$ every $f \in \mathbb{K}[X]$ has a unique NF $\quad\Leftrightarrow\quad \mathbb{K}[X] = I \oplus \mathbb{K}\,\mathrm{NF}(G)$

THE ALGORITHM

Input: a set of monic polynomials $f_1, \ldots, f_r \in \mathbb{K}[X]$

Output: a G.B. G of $I(f_1, \ldots, f_r)$

Init: $G := \{f_1, \ldots, f_r\}$ and $P := G \times G$

While $P \neq \emptyset$**:**

- ➔ remove p from P and reduce $\mathrm{spol}(p)$ into NF w.r.t. G
- ➔ add $\widehat{\mathrm{spol}(p)}$ to G and add all the corresponding pairs to P

Return G

Proof of termination: follows from Dickson's lemma

Proof of correctness: $G \subseteq I$ and every S-polynomials rewrites into 0

EXAMPLE

Input: $G := \{g_1, g_2\}$ with $g_1 := \mathbf{xy^2} + x$ and $g_2 := \mathbf{y^3} + (xy)/2 - 1/2$

$<$: the deglex order induced by $y < x$

While loop:

- ➔ $\mathrm{spol}(g_1, g_2) = -(x^2y)/2 + xy + x/2 \quad \rightsquigarrow \quad g_3 := \mathbf{x^2y} - 2xy - x \in G$
- ➔ $\mathrm{spol}(g_1, g_3) = 2xy^2 + x^2 + xy$ rewrites into $g_4 := \mathbf{x^2} + xy - 2x \in G$
- ➔ $\mathrm{spol}(g_2, g_3) = -(x^3y)/2 + x^2/2 - 2xy^3 - xy^2$ rewrites into 0
- ➔ $\mathrm{spol}(g_1, g_4) = xy^3 - 2xy^2 - x^2$ rewrites into 0
- ➔ $\mathrm{spol}(g_2, g_4) = xy^4 - (x^3y)/2 - xy^3 + x^2/2$ rewrites into 0
- ➔ $\mathrm{spol}(g_3, g_4) = xy^2 + x$ rewrites into 0

Return $\left\{xy^2 + x, \quad y^3 + (xy)/2 - 1/2, \quad x^2y - 2xy - x, \quad x^2 + xy - 2x\right\}$

1st Buchberger's criterion: if $\gcd(\mathrm{lm}(g), \mathrm{lm}(g')) = 1$, $\mathrm{spol}(g, g')$ rewrites into 0

➔ we may restrict the algorithm by computing S-pol. with nontrivial gcd

Alternatively: we obtain a linear adaptation of Knuth-Bendix algorithm

➔ with $g_1 := \mathbf{xy^2} + x$ and $g_2 := \mathbf{y^3} + (xy)/2 - 1/2$, we get

$g_3 := \mathbf{x^2y} - 2xy - x$ and $g_4 := \mathbf{x^2} + xy - 2x$ from

$$\begin{tikzcd}
 & xy^3 \arrow[dl, "yg_1"'] \arrow[dr, "xg_2"] & \\
-xy & & \frac{-x^2y+x}{2} \arrow[ll, dashed, "\frac{-g_3+x}{2}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x^2y^2 \arrow[dl, "xg_1"'] \arrow[dr, "yg_3"] & \\
-x^2 \arrow[dr, dashed, "-g_4"'] & & 2xy^2 + xy \arrow[dl, "2g_1+xy"] \\
 & xy - 2x &
\end{tikzcd}$$

Remark. Gröbner bases may be computed by Gaussian elimination

- ➔ consider a critical pair $l \leftarrow m \rightarrow r$
- ➔ reduce l and r into NF and store the reductions into a matrix M
- ➔ compute the row echelon form $\overline{M}$ of M by Gaussian elimination
- ➔ if some $\mathrm{lm}(\overline{M}_{i\bullet})$ does not belong to $\langle \mathrm{lm}(G) \rangle$, then add $\overline{M}_{i\bullet}$ to G

Illustration: consider $g_1 := xy^2 + x$ and $g_2 := y^3 + (xy)/2 - 1/2$

$$\begin{tikzcd} & xy^3 \arrow[dl, "yg_1"'] \arrow[dr, "xg_2"] & \\ -xy & & \frac{-x^2y+x}{2} \end{tikzcd}$$

$$\begin{array}{cc} \begin{array}{cccc} xy^3 & x^2y & xy & x \end{array} & \\ \left|\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right| & \begin{array}{c} yg_1 \\ xg_2 \end{array} \end{array}$$

By Gaussian elimination

Remark. Gröbner bases may be computed by Gaussian elimination

- ➔ consider a critical pair $l \leftarrow m \rightarrow r$
- ➔ reduce l and r into NF and store the reductions into a matrix M
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$$\begin{array}{ccccc} xy^3 & x^2y & xy & x & \\ \left|\begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1 \end{matrix}\right| & & & & \begin{matrix} yg_1 \\ 2(xg_2 - yg_1) \end{matrix} \end{array}$$

By **Gaussian elimination** we get

$$\mathbf{g_3 := x^2y - 2xy - x}$$

THE ALGORITHM

Input: a set of monic polynomials $f_1, \ldots, f_r \in \mathbb{K}[X]$

Output: a G.B. G of $I(f_1, \ldots, f_r)$

Init: $G := \{f_1, \ldots, f_r\}$ and $P := G \times G$

While $P \neq \emptyset$**:**

- ➔ remove from P a **selected** subset P' of P
- ➔ reduce the spol of elements of P' into normal form
- ➔ store all reductions into a matrix M
- ➔ compute the row echelon form $\overline{M}$ of M
- ➔ add to G each $\overline{M}_{i\bullet}$ with leading monomial not in $\langle \mathrm{lm}(G) \rangle$
- ➔ add the corresponding pairs to P

Return G

II. NONCOMMUTATIVE GRÖBNER BASES

OBJECTIVE: adapt G.B. theory

to the noncommutative framework

One need noncommutative adaptations of

- monomial orders ➔ definition of noncommutative G.B.
- polynomial reduction ➔ rewriting characterisation of noncommutative G.B.
- S-polynomials ➔ noncommutative Buchberger's/F4 procedures

We apply noncommutative G.B. to homological algebra ➔ Anick's resolution

Monomial order: **total**, **well-founded** order $<$ on noncommutative monomials $\langle X \rangle$, that is **admissible** *i.e.*,

$$\forall m, m', m_1, m_2 \in [X] : \quad m_1 < m_2 \;\Rightarrow\; m m_1 m' < m m_2 m'$$

Gröbner bases: a generating subset G of the ideal $I \subseteq \mathbb{K}\langle X \rangle$ s.t. $\mathrm{lm}(\mathbf{I}) = \langle \mathbf{lm(G)} \rangle$ (for a fixed monomial order $<$)

Polynomial reduction: given $G \subseteq \mathbb{K}\langle X \rangle$ and a monomial order $<$:

$$\lambda\Big(m\,\mathrm{lm}(g) m'\Big) + f \quad \to_G \quad \frac{\lambda}{\mathrm{lc}(g)}\Big(m\,\mathrm{rem}(g) m'\Big) + f$$

where $g \in G$, $\lambda \neq 0$, $m, m' \in \langle X \rangle$ **and** $m\,\mathrm{lm}(g) m' \notin \mathrm{supp}(f)$

Theorem. Let I be a noncommutative polynomial ideal, G be a generating set of I, and $<$ be a monomial order. Then,

$$G \text{ is a Gröbner basis of } I \qquad \Leftrightarrow \qquad \to_G \text{ is confluent}$$

Theorem. Let I be a noncommutative polynomial ideal, G be a generating set of I, and $<$ be a monomial order. Then,

$$G \text{ is a Gröbner basis of } I \quad \Leftrightarrow \quad \to_G \text{ is confluent}$$

Remark. If $\mathbf{A} = \mathbb{K}\langle X \mid R\rangle := \mathbb{K}\langle X\rangle/\langle R\rangle$ is an algebra and $<$ is a monomial order, then R is a Gröbner basis of $\langle R\rangle$ iff $\to_R$ is a confluent orientation of R.

In this case, $\mathbf{A}$ admits as a basis

$$\left\{ m \mod \langle R\rangle \mid \quad m \quad \text{is a normal form for} \quad \to_R \right\}$$

Two applications of:

"Gröbner bases $\leftrightarrow$ confluent orientations"

Ideal membership problem: given a G.B. G of I and $f \in \mathbb{K}\langle X \rangle$, how to decide $f \in I$?

➔ reduce f into normal form $\widehat{f}$ using G and test $\widehat{f} = 0$

➔ $\widehat{f}$ is independent from the reduction path!

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PBW theorem: let $\mathscr{L}$ be a Lie algebra and let X be a totally well-ordered basis of $\mathscr{L}$.

Then, the universal enveloping algebra $U(\mathscr{L})$ of $\mathscr{L}$ admits as a basis

$$\left\{x_1^{\alpha_1}\dots x_k^{\alpha_k} \mid \quad x_i < x_{i+1} \in X,\ \alpha_i \in \mathbb{N}\right\}$$

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Ideas of the proof:

➔ presentation of $U(\mathscr{L})$: $\mathbb{K}\langle X \mid yx - xy - [y,x], \quad x \neq y \in X \rangle$

➔ choice of orientation: $yx \to xy + [y,x]$, where $x < y$

➔ this orientation is confluent (equivalent to Jacobi identity)

➔ a basis of $U(\mathscr{L})$ is composed of NF monomials: $x_1^{\alpha_1} \dots x_k^{\alpha_k}$ s.t. $x_i < x_{i+1}$

S-polynomials

Ambiguities of $G \subseteq \mathbb{K}\langle X\rangle$: tuples $\mathfrak{a} = (w_1, w_2, w_3, g, g')$ such that

- $w_1, w_2, w_3 \in \langle X\rangle$ with $w_2 \neq 1$ and $g, g' \in G$
- one of the following two conditions holds
  - ➔ $w_1 w_2 = \mathrm{lm}(g)$ and $w_2 w_3 = \mathrm{lm}(g')$ (overlapping)
  - ➔ $w_1 w_2 w_3 = \mathrm{lm}(g)$ and $w_2 = \mathrm{lm}(g')$ (inclusion)

S-polynomials: $\mathrm{spol}(\mathfrak{a}) = g w_3 - w_1 g'$ if $\mathfrak{a}$ is an overlapping

$\mathrm{spol}(\mathfrak{a}) = w_1 g w_3 - g'$ if $\mathfrak{a}$ is an inclusion

Proposition: G is a noncommutative G.B. iff every spol rewrites into 0

Completion procedures: adaptations of Buchberger's and F_4 procedures

Fix $G \subseteq \mathbb{K}\langle X\rangle$ and a monomial order

Definition: Anick's n-chains and their tails are defined by induction

- the unique (-1)-chain is 1, which is its own tail
  0-chains are elements of X, which are their own tails

- if $n \geq 1$: a n-chain with tail t is a monomial mt such that

  i. m is a $(n-1)$-chain with tail t'

  ii. t is a normal form w.r.t. G

  iii. $t't$ is uniquely reducible, on its right

Example: if $\mathrm{lm}(G) = \{xyx, yxy\}$

0-chains: x and y **1-chains:** x**yx** and y**xy** **2-chains:** xyx**y** and yxy**x**

3-chains: $xyxy$**xy** and $yxyx$**yx** **4-chains:** $xyxyxy$**x** and $yxyxyx$**y**

Framework:

We fix: $\mathbf{A} = \mathbb{K}\langle X \mid R\rangle \overset{\varepsilon}{\to} \mathbb{K}$ (with $\ker(\varepsilon) = \langle X\rangle$) and a monomial order $<$

Assumption: **R is a reduced noncommutative Gröbner basis** of $\langle R\rangle$

Construction of Anick's resolution: main steps

➔ consider the free (left $\mathbf{A}-$)modules $\mathbf{A}\langle \mathcal{C}_n\rangle$ generated by n-chains

$(\mathbf{A}\langle \mathcal{C}_{-1}\rangle \simeq \mathbf{A}, \qquad \mathbf{A}\langle \mathcal{C}_0\rangle = \mathbf{A}\langle X\rangle, \qquad \mathbf{A}\langle \mathcal{C}_1\rangle \simeq \mathbf{A}\langle R\rangle)$

➔ boundaries ∂_n are constructed simultaneously with the contracting homotopy ι_n

they satisfy the identities: $\quad \partial_n \circ \partial_{n+1} = 0 \quad$ and $\quad \partial_{n+1} \circ \iota_n = \mathrm{id}_{\ker(\partial_n)}$

$$
\begin{tikzcd}
\ldots \arrow[r] & \mathbf{A}\langle \mathcal{C}_n\rangle \arrow[r, bend left, "\partial_n"] & \mathbf{A}\langle \mathcal{C}_{n-1}\rangle \arrow[l, bend left, "\iota_{n-1}"] \arrow[r] & \ldots \arrow[r] & \mathbf{A}\langle \mathcal{C}_1\rangle \arrow[r, bend left, "\partial_1"] & \mathbf{A}\langle \mathcal{C}_0\rangle \arrow[l, bend left, "\iota_0"] \arrow[r, bend left, "\partial_0"] & \mathbf{A}\langle \mathcal{C}_{-1}\rangle \arrow[l, bend left, "\iota_{-1}"] \arrow[r, "\varepsilon"] & \mathbb{K} \arrow[r] & 0
\end{tikzcd}
$$

Required relations: $\partial_n \circ \partial_{n+1} = 0$ and $\partial_{n+1} \circ \iota_n = \mathrm{id}_{\ker(\partial_n)}$

$$\begin{tikzcd}
\mathbf{A}\langle \mathrm{lm}(R) \rangle \arrow[r, bend left, "\partial_1"] & \mathbf{A}\langle X \rangle \arrow[l, bend left, "\iota_0"] \arrow[r, bend left, "\partial_0"] & \mathbf{A} \arrow[l, bend left, "\iota_{-1}"] \arrow[r, "\varepsilon"] & \mathbb{K} \arrow[r] & 0
\end{tikzcd}$$

∂_0 **and** ι_{-1}**:** $\partial_0([x]) := \overline{x}$ and $\iota_{-1}(\overline{mx}) := \overline{m}.[x]$ $(mx \in \mathrm{NF})$

➔ $\ker(\varepsilon) = \mathrm{im}(\partial_0)$ and $\partial_0 \circ \iota_{-1} = \mathrm{id}_{\ker(\varepsilon)}$

∂_1 **and** ι_0**:** $\partial_1([\mathrm{lm}(g)]) := \overline{m}.[x] - \sum \lambda_i \overline{m_i}.[x_i]$ where $g = mx - \sum \lambda_i m_i x_i$

➔ $\partial_0 \circ \partial_1 = 0$ since $\partial_1[\mathrm{lm}(g)] = \overline{m}.[x] - \iota_{-1} \circ \partial_0(\overline{m}.[x])$

$\forall h \in \ker(\partial_0)$ with leading term $\overline{m}.[x]$, there is a factorisation $mx = m' \,\mathrm{lm}(g)$

$$\iota_0(h) := \overline{m'}.[\mathrm{lm}(g)] + \iota_0\Big(\big(h - \partial_1\big(\overline{m'}.[\mathrm{lm}(g)]\big)\Big)$$

➔ $\partial_1 \circ \iota_0 = \mathrm{id}_{\ker(\partial_0)}$ is proven by induction

Required relations: $\partial_n \circ \partial_{n+1} = 0$ and $\partial_{n+1} \circ \iota_n = \mathrm{id}_{\ker(\partial_n)}$

$$\begin{tikzcd} \mathbf{A}\langle \mathrm{lm}(R)\rangle \arrow[r, bend left, "\partial_1"] & \mathbf{A}\langle X\rangle \arrow[l, bend left, "\iota_0"] \arrow[r, bend left, "\partial_0"] & \mathbf{A} \arrow[l, bend left, "\iota_{-1}"] \arrow[r, "\varepsilon"] & \mathbb{K} \arrow[r] & 0 \end{tikzcd}$$

∂_0 and ι_{-1}: $\partial_0([x]) := \overline{x}$ and $\iota_{-1}(\overline{mx}) := \overline{m}.[x]$ $(mx \in \mathrm{NF})$

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Required relations: $\partial_n \circ \partial_{n+1} = 0$ and $\partial_{n+1} \circ \iota_n = \mathrm{id}_{\ker(\partial_n)}$

$$\begin{tikzcd}
\mathbf{A}\langle\mathcal{C}_{n+1}\rangle \arrow[r, bend left, "\partial_{n+1}"] & \mathbf{A}\langle\mathcal{C}_n\rangle \arrow[l, bend left, "\iota_n"] \arrow[r, bend left, "\partial_n"] & \mathbf{A}\langle\mathcal{C}_{n-1}\rangle \arrow[l, bend left, "\iota_{n-1}"]
\end{tikzcd}$$

$\mathbf{A}\langle\mathcal{C}_{n+1}\rangle \xrightarrow{\partial_{n+1}} \mathbf{A}\langle\mathcal{C}_n\rangle$: $\partial_{n+1}([m \mid t]) := \overline{m}.[t] - \iota_{n-1} \circ \partial_n(\overline{m}.[t])$

➔ $\partial_n \circ \partial_{n+1} = 0$ (using the induction hypothesis $\partial_n \circ \iota_{n-1} = 0$)

$\mathbf{A}\langle\mathcal{C}_n\rangle \xrightarrow{\iota_n} \mathbf{A}\langle\mathcal{C}_{n+1}\rangle$: $\forall h \in \ker(\partial_n)$ with leading term $\overline{m}.[c]$, $mc = m'c'$, with $c' \in \mathcal{C}_{n+1}$

$$\iota_n(h) := \overline{m'}.[c'] + \iota_0\Big((c - \partial_1(\overline{m'}.[c'])\Big)$$

➔ $\partial_{n+1} \circ \iota_n = \mathrm{id}_{\ker(\partial_n)}$ is proven by induction

Using the Anick'resolution, we can prove:

- if $\mathbf{A} = \mathbb{K}\langle X \mid R \rangle$ is a monomial algebra, *i.e.* $R \subseteq \langle X \rangle$, then

$$\mathrm{Tor}^{\mathbf{A}}(\mathbb{K}, \mathbb{K}) = \bigoplus_{n} \mathbb{K}\mathcal{C}_n$$

- if $\mathbf{A}$ is presented by a quadratic Gröbner basis, then it is Koszul

- if $\mathbf{A}$ is presented by an N-homogeneous Gröbner basis and satisfies the extra-condition, then it is N-Koszul

IV. REDUCTION OPERATORS

A brief overview on reduction operators

Bergman, 1978: formalism for rewriting noncommutative polynomials

Berger 1998, 2001: lattice characterisation of homogeneous G.B.

applied to Koszul duality

C. 2016, 2018: **lattice characterisations of confluence and completion**

with the following applications

- ➔ constructive proof of Koszulness
- ➔ lattice formulation of the noncommutative F_4 completion procedure
- ➔ computation of syzygies and detection of useless critical pairs

Functional representation of rewriting strategies

Example: $yy \to yx \quad \rightsquigarrow$ **left**/**right** reduction operators on 3 letter words

$$\begin{tikzcd} & yyy \arrow[dl, "\mathsf{L}"'] \arrow[dr, "\mathsf{R}"] & \\ yxy & & yyx \arrow[dl, "\mathsf{L}"] \\ & yxx & \end{tikzcd}$$

Properties of L and R: they are linear projectors of $\mathbb{K}X^{(3)}$ (or $\mathbb{K}\langle X\rangle$) and compatible with the deglex order induced by $x < y$

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Properties of L and R: they are linear projectors of $\mathbb{K}X^{(3)}$ (or $\mathbb{K}\langle X\rangle$) and compatible with the deglex order induced by $x < y$

Definition: a **reduction operator** on a vector space V equipped with a well-ordered basis $(G, <)$ is a linear projector of V s.t.

$$\forall g \in G: \qquad T(g) = g \quad \text{or} \quad \operatorname{lm}(T(g)) < g$$

Remark. Finite dimensional restrictions of R.O. admit matrix representations, *e.g.*,

$$\begin{tikzcd} & yyy \arrow[dl, "\mathbf{L}"'] \arrow[dr, "\mathbf{R}"] & \\ yxy & & yyx \arrow[dl, "\mathbf{L}"] \\ & yxx & \end{tikzcd}$$

The matrix representations of **L** and **R** are

$$\mathbf{L} = \begin{array}{c} \begin{array}{cccc} yxx & yxy & yyx & yyy \end{array} \\ \left|\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right| \end{array} \qquad \mathbf{R} = \begin{array}{c} \begin{array}{cccc} yxx & yxy & yyx & yyy \end{array} \\ \left|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right| \end{array}$$

Proposition: the kernel map induces a bijection between R.O. and subspaces

$$\ker : \quad \left\{ \textbf{reduction operators on } V \right\} \quad \leftrightarrow \quad \left\{ \textbf{subspaces of } V \right\}$$

In particular, reduction operators admit the following lattice operations

- ➔ $T_1 \preceq T_2$ iff $\ker(T_2) \subseteq \ker(T_1)$
- ➔ $T_1 \wedge T_2$ is the reduction operator with kernel $\ker(T_1) + \ker(T_2)$
- ➔ $T_1 \vee T_2$ is the reduction operator with kernel $\ker(T_1) \cap \ker(T_2)$

Moreover, $T_1 \wedge T_2$ computes minimal normal forms

$$\begin{tikzcd}
& yyy \arrow[dl, "L"'] \arrow[dr, "R"] & \\
yxy & & yyx \arrow[dl, "L"] \\
& yxx &
\end{tikzcd}$$

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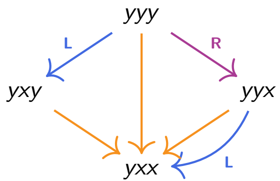

Computing lower bound using Gaussian elimination

Example: consider

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \ker(\mathbf{L} \wedge \mathbf{R}) &= \ker(\mathbf{L}) + \ker(\mathbf{R}) = \mathbb{K}\{yyx - yxx, yyy - yxy, yyy - yyx\} \\ &= \mathbb{K}\{\mathbf{yxy\text{-}yxx}, \quad \mathbf{yyx\text{-}yxx}, \quad \mathbf{yyy\text{-}yxx}\} \end{aligned}$$

Hence

$$\mathbf{L} \wedge \mathbf{R} = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Lemma/Definition. For a familly F of R.O., we have

$$\operatorname{im}(\wedge F) \subseteq \bigcap_{T \in F} \operatorname{im}(T) \quad \rightsquigarrow \quad \bigcap_{T \in F} \operatorname{im}(T) = \operatorname{im}(\wedge F) \oplus \mathbb{K}\operatorname{obs}(F)$$

Illustration. Consider

$$\begin{tikzcd}
 & yyy \arrow[dl, "L"'] \arrow[dr, "R"] \arrow[dd] & \\
yxy \arrow[dr] & & yyx \arrow[dl] \arrow[dl, bend left, "L"] \\
 & yxx &
\end{tikzcd}$$

$$\operatorname{im}(\mathbf{L}) \cap \operatorname{im}(\mathbf{R}) = \operatorname{im}(\mathbf{L} \wedge \mathbf{R}) \oplus \mathbb{K}\{yxy\} \quad \Rightarrow \quad \operatorname{obs}(\mathbf{L}, \mathbf{R}) = \mathbb{K}\{yxy\}$$

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Remark. $(\mathbf{L}, \mathbf{R})$ is **completed by the operator** mapping any obstruction to its image by the lower bound

Theorem. Let F be a family of reduction operators and $\to_F$ be the induced rewriting relation on V. Then, $\to_F$ is confluent if and only if

$$\operatorname{im}(\wedge F) \quad = \quad \bigcap_{T \in F} \operatorname{im}(T)$$

Moreover, if $\to_F$ is not confluent, then F is completed by

$$C(F) \quad := \quad \wedge F \vee \left(\vee \overline{F}\right)$$

where

$$\vee \overline{F} \quad := \quad \ker^{-1}\left(\bigcap_{T \in F} \operatorname{im}(T)\right)$$