A geometric stabilization of planar switched systems

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Continuous-time switched linear systems

A (continuous-time) switched linear system is given by

$$\dot{x}(t) = A_{\sigma(t)}x(t), \qquad t \in \mathbb{R}_{\geq 0}, \quad x(0) \in \mathbb{R}^n$$

where

- $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ represents the state variable, $x(0)$ is the initial state
- $A_1, \cdots, A_p \in \mathbb{R}^{n \times n}$ are subsystems
- $\sigma : \mathbb{R}_{\geq 0} \to \{A_1, \cdots, A_p\}$ is the switching signal

Problem

Construct a **stabilizing signal**:

➔ the corresponding trajectory converges to 0 with exponential decay

Existing stability/stabilization analysis methods

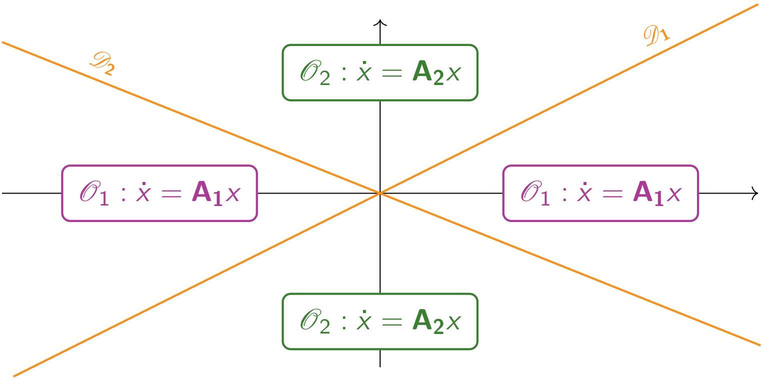
- Traces and determinants (Balde and *al.*)
- Joint spectral radius (Blondel)
- Lie algebraic conditions (Liberzon, Gurvitz)
- Set theoretic approach (Megretski, Kruszewski, Guerra)
- **Stability analysis under restricted switching** (Lin-Antsalkis)

A CANDIDATE FOR STABILIZATION

Assumptions: 2 modes and 2 states → fix $\mathbf{A_1}$, $\mathbf{A_2} \in \mathbb{R}^{2\times 2}$

The candidate: $\sigma : \mathbb{R}_{\geq 0} \to \{\mathbf{A_1}, \mathbf{A_2}\}$ defined by

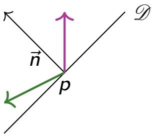
- choose two lines $\mathscr{D}_1$ and $\mathscr{D}_2$ and label adjacent regions by $\mathscr{O}_1$ and $\mathscr{O}_2$
- define $\sigma(t) = \mathbf{A_i}$ if $x(t) \in \mathscr{O}_i$ and $\sigma(t) = \mathbf{A_1}$ or $\mathbf{A_2}$ if $x(t) \in \mathscr{D}_1 \cup \mathscr{D}_2$



Sliding motions

Let $\mathscr{D}$ be a line in $\mathbb{R}^2$, let $p \in \mathscr{D}$ and let $\vec{n}$ be a unit vector normal to $\mathscr{D}$ at p

Definition 1: We say that no sliding motion occurs at p if $(\mathbf{A}_1\vec{n})^T(\mathbf{A}_2\vec{n}) > 0$
(equivalently: $\mathbf{A}_1$ and $\mathbf{A}_2$ point in the same half-plane at p)



Definition 2: $\mathscr{D}$ induces no sliding motion for $(\mathbf{A}_1, \mathbf{A}_2)$ if no sliding motion occurs at any point

Our contributions

Let $\sigma : \mathbb{R}_{\geq +} \to \{\mathbf{A}_1, \mathbf{A}_2\}$ be a switching candidate induced by $\mathscr{D}_1$ and $\mathscr{D}_2$

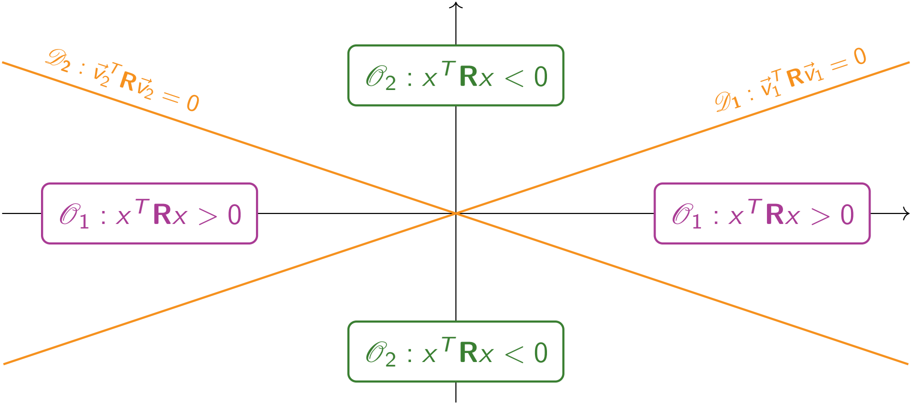
Question: if $\mathscr{D}_1$ and $\mathscr{D}_2$ induce no sliding motion for $(\mathbf{A}_1, \mathbf{A}_2)$, is σ stabilizing?

Our results: 2 sufficient conditions based on Lyapunov functions design

➔ we present the first condition and sketch the second one

Algebraic reformulation of the problem

- choose an orientation vector $\vec{v}_i$ of $\mathscr{D}_i$ and define $\mathbf{R} \in \mathbb{R}^{2\times 2}$ by $\vec{v}_i^T \mathbf{R} \vec{v}_i = 0$ (without restriction, $\mathbf{R}$ is assumed to be symmetric with determinant < 0)
- the region $\mathscr{O}_1$ (resp., $\mathscr{O}_2$) are points $x \in \mathbb{R}^2$ s.t. $x^T \mathbf{R} x > 0$ (resp., $x^T \mathbf{R} x < 0$)



Theorem

Let $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{R}^{2\times 2}$, let $\mathbf{R}$ be a symmetric matrix with negative determinant, let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ such that $\vec{v}_i^T \mathbf{R} \vec{v}_i = 0$ and assume that $\mathscr{D}_1$ and $\mathscr{D}_2$ induce no sliding motion. If the following LMIs problem admits a solution:

$$\exists \tau_1, \tau_2 \geq 0 \text{ and } P_1 = P_1^T, P_2 = P_2^T \text{ s.t.}$$

(1): $P_1 - \mathbf{R} > 0, \quad P_2 + \mathbf{R} > 0$

(2): $-(\mathbf{A}_1^T P_1 + P_1 \mathbf{A}_1) - \tau_1 \mathbf{R} > 0, \qquad -(\mathbf{A}_2^T P_2 + P_2 \mathbf{A}_2) + \tau_2 \mathbf{R} > 0$

(3): $\vec{v}_1^T \ (P_1 - P_2)\ \vec{v}_1 = 0, \quad \vec{v}_2^T \ (P_1 - P_2)\ \vec{v}_2 = 0$

then, $\sigma : \mathbb{R}_{\geq 0} \to \{\mathbf{A}_1, \mathbf{A}_2\}$ defined by $\sigma(t) = \mathbf{A}_1$ (resp., $\mathbf{A}_2$) if $x^T \mathbf{R} x > 0$ (resp., < 0) is a stabilizing signal.

Ideas of the proof : $V : \mathbb{R}^2 \to \mathbb{R}$ defined by $V(x) = x^T P_i x$ if $x \in \mathscr{O}_i$ is

- well-defined (from (**3**))
- positive and strictly decreasing along trajectories (from (**1**) and (**2**))

A numerical example

From Lin-Antsalkis consider the following two modes

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 10 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{A}_2 = \begin{pmatrix} 1.5 & 2 \\ -2 & -0.5 \end{pmatrix}$$

and the stabilizing switching induced by $\mathscr{D}_1$ and $\mathscr{D}_2$ with orientation vectors

$$\vec{v}_1 = \begin{pmatrix} 1 & 0.3 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 & 0.11 \end{pmatrix}$$

The corresponding matrix is

$$\mathbf{R} = \begin{pmatrix} -0.033 & -0.095 \\ -0.095 & 1 \end{pmatrix}$$

and the LMIs problem has the following solution

$$\tau_1 = 1.8890, \quad \tau_2 = 1.3550, \quad P_1 = \begin{pmatrix} 0.0229 & -0.1376 \\ -0.1376 & 3.9156 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0.1424 & 0.2065 \\ 0.2065 & 0.2940 \end{pmatrix}$$

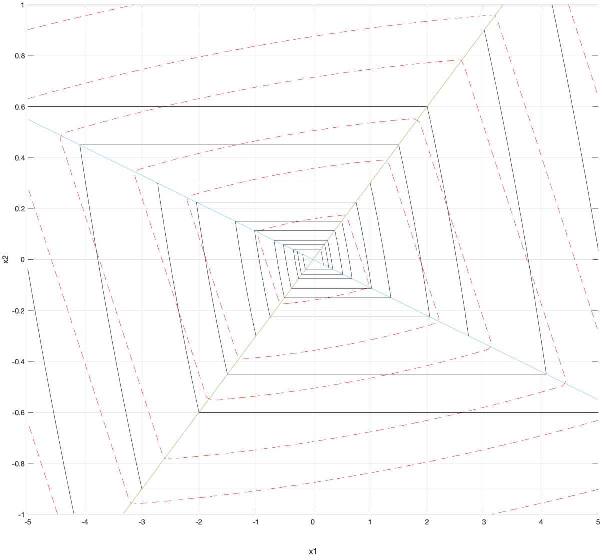
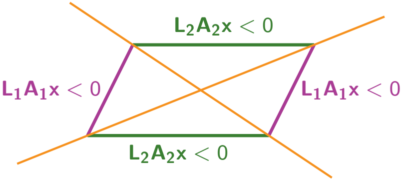


Figure: trajectories converging to the origin are depicted in solid black lines and level sets of the Lyapunov function (dashed red curves) form "contracting parallelograms"

Geometric reformulation of the problem

A set $\mathscr{P}$ of parallelogram is said to be **contracting** for $(\mathscr{D}_1, \mathscr{D}_2)$ if

- the sides of each parallelogram in $\mathscr{P}$ are oriented by the same vectors $\mathbf{L_1}$ and $\mathbf{L_2}$
- $\mathscr{D}_1$ and $\mathscr{D}_2$ are the diagonals of each parallelogram in $\mathscr{P}$
- trajectories are decreasing along sides of parallelograms in $\mathscr{P}$



Theorem

Assume that $\mathscr{D}_1$ and $\mathscr{D}_2$ induce no sliding motions and that a contracting set of parallelograms exists. The switching signal defined by $\mathscr{D}_1$ and $\mathscr{D}_2$ is stabilizing.

Conclusion and perspectives

- We provided two methods for checking stabilization of planar switched systems induced by lines passing through the origin
- Our methods are based on an algebraic reformulation of the problem and constructions of Lyapunov functions
- Further works include: construction of stabilizing signal, robustness, allow sliding motions

THANK YOU FOR LISTENING!