Compatible rewriting of noncommutative polynomials for proving operator identities

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JOHANNES KEPLER UNIVERSITÄT LINZ

Der Wissenschaftsfonds.

Proving operator identities

Objective: formally prove operator identities

- ▷ operators are expressible in terms of basic operators
- ▷ "forgetting" the analytic meaning by replacing basic operators by symbols

Prove new identities $\rightsquigarrow$ establish equalities in suitable algebraic structures, *e.g.*,

- ▷ linear P.D.E.'s with constant/polynomial coeff. $\rightsquigarrow$ polynomial/Weyl algebras
- ▷ integro-diff. systems with smooth unknown functions $\rightsquigarrow$ tensor algebras
- ▷ other systems with mixed operations $\rightsquigarrow$ Ore algebras/extensions, tensor rings

Prove algebraic equalities $\rightsquigarrow$ use rewriting theory

- ▷ *e.g.*, (adaptations of) Gröbner/Janet bases, tensor reduction systems, ...
- ▷ simplify a syntactic expression into an equivalent one, *e.g.*,

$$\partial \circ \int = \mathbf{Id} : \qquad A \circ \partial \circ \int \circ B \longrightarrow A \circ B$$

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Additional task

Take compatibility conditions into account

- ▷ multiplication is not defined everywhere, *e.g.*, matrices
- ▷ composition depends on domains and codomains

$$e.g., \qquad \partial : C^{k+1}(I) \to C^k(I), \qquad \int : C^k(I) \to C^{k+1}(I)$$

Existing method: based on quiver representation (Hossein Poor, R., R., arXiv:1910.06165)

- ▷ requires to work with "uniformly compatible" polynomials

Our contributions

Theoretical part: extend the quiver approach to prove more identities

- ➔ based on Q-consequences

Algorithmic part: compute Q-consequences using rewriting

- ➔ restrictions on the computations with G.B.

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Proof of operator identities

Given: basic operators satisfying identities, *e.g.*,

$$\partial(f) := f', \qquad \int(f) := \int_{x_0}^{x} f(t)dt, \qquad \mathbf{Eval}(f) := f(x_0)$$

are s.t.

$$\int \circ \partial = \mathbf{Id} - \mathbf{Eval}, \qquad \partial \circ \int = \mathbf{Id}$$

i.e., $\forall f :$ $\int_{x_0}^{x} f'(t)dt = f(x) - f(x_0)$, $\left(\int_{x_0}^{x} f(t)dt\right)' = f(x)$

Objective: prove new identities using symbolic methods, *e.g.*,

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Formal computations with noncommutative polynomials

Example:

$$\partial, \quad \int, \quad \textbf{Eval} \quad \text{and} \quad \int \circ \partial - \textbf{Id} + \textbf{Eval} = 0, \quad \partial \circ \int - \textbf{Id} = 0$$

Polynomial translation:

$$\mathbb{K}\langle d, i, e\rangle \quad \ni \quad id - 1 + e, \; di - 1$$

New identity:

$$ei = (id - 1 + e)i - i(di - 1)$$

Additionally: check compatiblity of cofactor decomposition with domains and codomains

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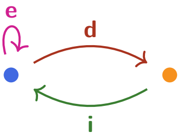
Quivers represented by operators



Def.: consider a labelled quiver Q (one letter may label multiple edges)

- $f \in \mathbb{K}\langle X\rangle$ is a Q-consequence of $F \subseteq \mathbb{K}\langle X\rangle$ if it admits a compatible decomposition

Ex. of a Q-csq.: $ei = (id - 1 + e)i - i(di - 1) = idi - i + ei - idi + i$

each monomial labels a path $\bullet \xrightarrow{*} \bullet$

Theorem

If all elements of realizations of F are zero and if f is a Q-consequence of F, then all realizations of f are zero

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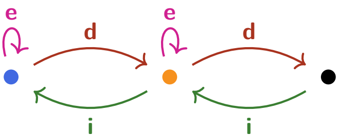

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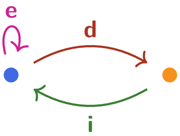

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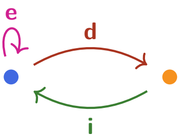

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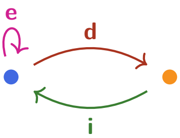

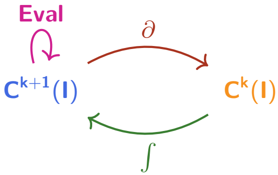

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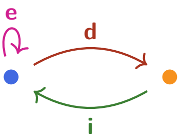

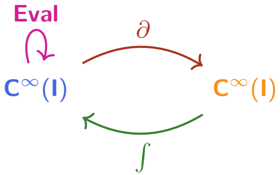

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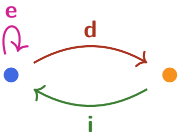

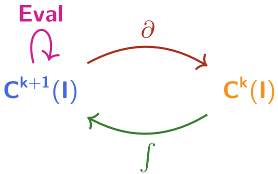

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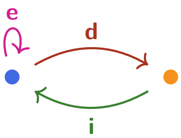

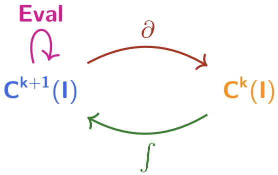

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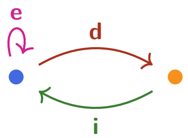

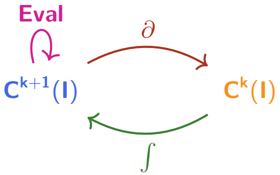

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Illustrating example

Consider the inhomogeneous linear O.D.E.

$$y''(x) + A_1(x)y'(x) + A_0(x)y(x) = r(x) \tag{1}$$

Assumption: (1) can be factored into the 1st order equations

$$y'(x) - B_2(x)y(x) = z(x) \quad \text{and} \quad z'(x) - B_1(x)z(x) = r(x)$$

General solution: given by

$$y(x) = H_2(x) \int_{x_2}^{x} H_2(t)^{-1} H_1(t) \int_{x_1}^{t} H_1(u)^{-1} r(u)\, du\, dt \tag{2}$$

where $H_i(x)$ is s.t. $H_i'(x) - B_i(x)H_i(x) = 0$ and $H_i(x)^{-1}$ exists

Illustration of the theorem: formally prove that (2) is a solution of (1)

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Assumptions: prove that

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is solution of

$$\big((\partial - B_1) \circ (\partial - B_2)\big)(y(x)) = r(x)$$

where H_i is s.t. $H_i'(x) - B_i(x)H_i(x) = 0$ and $H_i(x)^{-1}$ exists

Algebraic part: $X := \{h_1, h_2, b_1, b_2, \tilde{h}_1, \tilde{h}_2, i, d\}$, $F := \{f_1, \ldots, f_5\} \subset \mathbb{K}\langle X \rangle$, where

$$f_1 := dh_1 - h_1 d - b_1 h_1, \qquad f_2 := dh_2 - h_2 d - b_2 h_2,$$

$$f_3 := h_1 \tilde{h}_1 - 1, \qquad f_4 := h_2 \tilde{h}_2 - 1, \qquad f_5 := di - 1$$

Objective: prove that f is a Q-consequence of F, where

$$f := (d - b_1)(d - b_2) h_2 i \tilde{h}_2 h_1 i \tilde{h}_1 - 1$$

Represented quiver: we need 2nd order derivative/integration and regularity assumptions

Fact: $F := \{f_1, \dots, f_5\} \quad \Rightarrow \quad f \xrightarrow{*}_F 0$ using an orientation of f_i's

$$dh_1 \longrightarrow h_1 d + b_1 h_1, \qquad dh_2 \longrightarrow h_2 d + b_2 h_2,$$
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and keeping track of cofactors, we get

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By a case analysis: (3) proves that f is a Q-consequence of F

First method for proving Q-consequences

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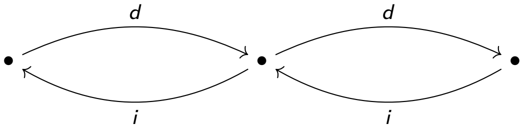

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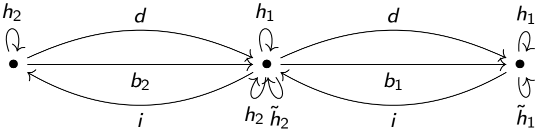

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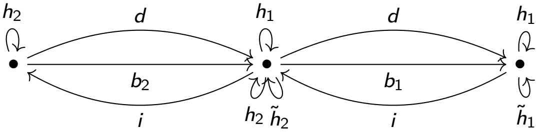

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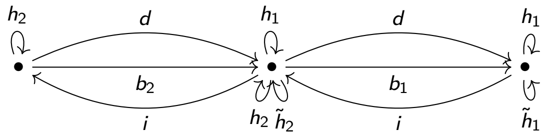

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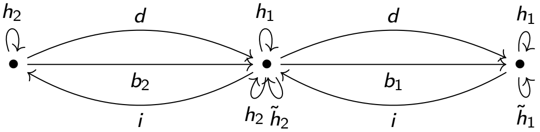

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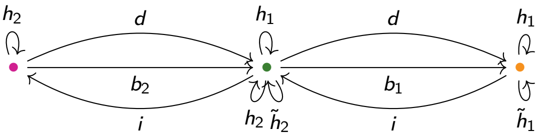
Second method for proving Q-consequences

Restrict to rew. steps s.t.

We only use "valid" computations

Compatible rewriting rules

Signatures: $\sigma(dh_2) = \{\bullet \overset{*}{\to} \bullet, \quad \bullet \overset{*}{\to} \bullet\}$, $\quad \sigma(h_2 d) = \{\bullet \overset{*}{\to} \bullet\}$, $\quad \sigma(b_2 h_2) = \{\bullet \overset{*}{\to} \bullet\}$



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- ▷ $dh_2 \to h_2 d + b_2 h_2$ is not compatible (1st method involved invalid computations)
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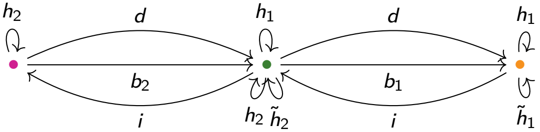

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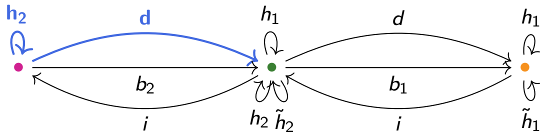

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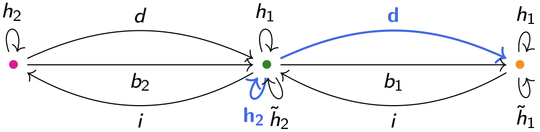

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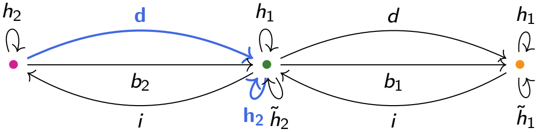

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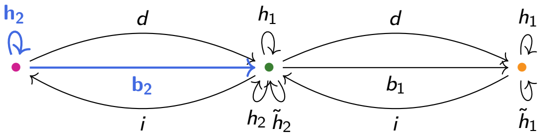

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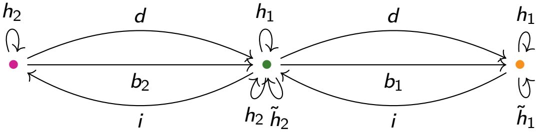

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Theorem

Let Q be a quiver labelled by X, let $F \subset \mathbb{K}\langle X\rangle$ and let $f \in \mathbb{K}\langle X\rangle$. Assume that each rew. rule is Q-compatible and $f \xrightarrow{*}_F 0$. Then,

$$f \text{ is compatible with } Q \quad \Leftrightarrow \quad f \text{ is a } Q\text{-consequence}$$

Summary of the 2nd method for proving Q-consequences

Using the Theorem:

- representation(s) of the quiver $\rightsquigarrow$ map any polynomial to the operator(s) it represents
- elements of F $\rightsquigarrow$ polynomial expressions of known operator identities
- f $\rightsquigarrow$ polynomial expression of the identity we wish to prove
- $f \xrightarrow{*}_F 0$ with compatible rew. rules only $\rightsquigarrow$ the identity is proven

Completion

Motivating example: consider as previously $F := \{f_1, \dots, f_5\}$ and f

- ▷ for the deglex order s.t. $b_1, h_1 < d < b_2 < h_2$, we get the compatible rew. rules

$$dh_1 \to h_1 d + b_1 h_1, \quad h_2 d \to dh_2 - b_2 h_2, \quad h_1\tilde{h}_1 \to 1, \quad h_2\tilde{h}_2 \to 1, \quad di \to 1$$

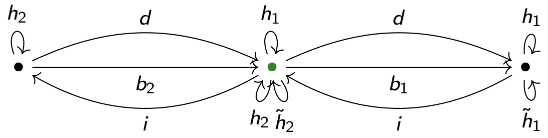
- ▷ problem: f does not rewrite into 0

$\rightsquigarrow$ we need a compatible completion procedure

Adaptation of the Buchberger's proc.: the compatible monomial $h_2 di$ induces

$$\mathrm{SP} = dh_2 i - b_2 h_2 i - h_2, \quad \mathrm{LM(SP)} = b_2 h_2 i$$

- ▷ $\sigma(b_2 h_2 i) = \{\bullet \xrightarrow{*} \bullet\} \subseteq \sigma(dh_2 i) \cap \sigma(h_2)$ $\rightsquigarrow$ we keep $f_6 := dh_2 i - b_2 h_2 i - h_2$



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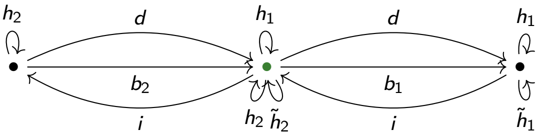

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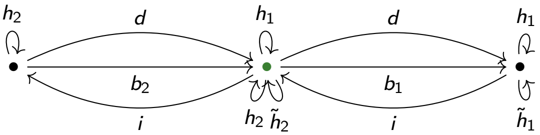

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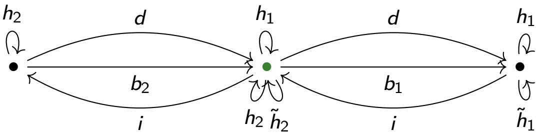

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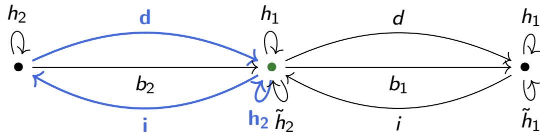

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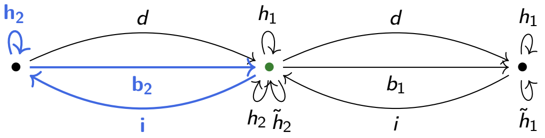

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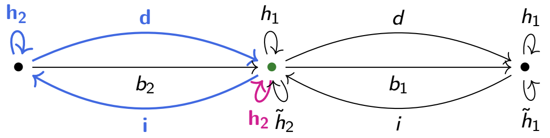

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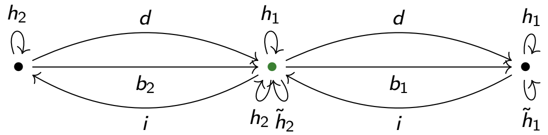

Many proofs at once

Using completion: letting $G := F \cup \{f_6\}$, we have $f \xrightarrow{*}_G 0$

From the compatibility theorem: for all representations φ of Q

$$\forall g \in F: \quad \varphi(g) = 0 \qquad \Rightarrow \qquad \varphi(f) = 0$$

Consequences: let us consider the linear O.D.E.

$$y''(x) + A_1(x)y'(x) + A_0(x)y(x) = r(x) \tag{4}$$

where

- $A_0,\ A_1$ are functions of class C^k
- r is a function of class C^k

If (4) may be factored into 1st order O.D.E.'s with homogeneous invertible sol. H_i,

$$y(x) = H_2(x) \int_{x_2}^{x} H_2(t)^{-1} H_1(t) \int_{x_1}^{t} H_1(u)^{-1} r(u)\, du\, dt$$

is a function of class C^{k+2} solution of (4)

Many proofs at once

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$$y''(x) + A_1(x)y'(x) + A_0(x)y(x) = r(x) \tag{4}$$

where

- $A_0, \; A_1$ are analytic functions
- r is an analytic function

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Consequences: let us consider the linear O.D.E.

$$y''(x) + A_1(x)y'(x) + A_0(x)y(x) = r(x) \tag{4}$$

where

- $A_0,\ A_1$ are **$n \times n$ matrices of functions of class C^k**
- r is a **vector of n functions of class C^k**

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$$y(x) = H_2(x) \int_{x_2}^{x} H_2(t)^{-1} H_1(t) \int_{x_1}^{t} H_1(u)^{-1} r(u)\, du\, dt$$

is a **vector of n functions of class C^{k+2}** solution of (4)

Many proofs at once

Using completion: letting $G := F \cup \{f_6\}$, we have $f \overset{*}{\to}_G 0$

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$$y''(x) + A_1(x)y'(x) + A_0(x)y(x) = r(x) \tag{4}$$

where

- ▷ $A_0,\ A_1$ are $\cdots$
- ▷ r is a $\cdots$

If (4) may be factored into 1st order O.D.E.'s with homogeneous invertible sol. H_i,

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is a $\cdots$ solution of (4)

Summary

Our contributions

- ▷ we develop an approach based on Q-consequences to formally prove identities
- ▷ we provided a method for computing Q-consequences using rewriting

Implementation:

- ▷ *Mathematica* package `OperatorGB` (by Clemens Hofstadler)
- ▷ link: `http://gregensburger.com/softw/OperatorGB/`

THANK YOU FOR YOUR ATTENTION!

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