

Computation of Koszul homology and application to partial differential systems

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Séminaire Calcul Formel

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I. Motivations

- ▷ Algebraic analysis and formal theory of PDEs
- ▷ Formal integrability
- ▷ 2-acyclicity and Koszul homology

II. Computation of Koszul homology

- ▷ Symbol module
- ▷ Koszul homology of the symbol module
- ▷ 2-acyclicity criterion

III. Examples

- ▷ Constant coefficients
- ▷ Parametric coefficients
- ▷ Polynomial coefficients

IV. Conclusion and perspectives

I. MOTIVATIONS

Consider a PDEs system

$$\Sigma : F^p \left(x^i, \frac{\partial^{|\mu|} u^j}{\partial x^\mu} \right) = 0$$

→ $x^1, \dots, x^n, u^1, \dots, u^p$: (in)dependent variables

→ F^1, \dots, F^q : equations of the system

Study of Σ

Analytical approach

Objective: prove existence and uniqueness of solutions

→ what type of solution?
(classical, weak, distributions)

Algorithmic side: approximate solutions by numerical schemes

Algebraic approach

Objective: investigate algebraic structures of the solution space
→ symbolic solutions (formal power series, Laplace inversion)

Algorithmic side: compute exact solutions

Algebraic analysis:

Apply tools from algebra and algebraic geometry to PDEs
(D -modules, homological algebra, sheaf theory)

Module of a linear system $Ru = 0$: $\mathcal{M} := \text{coker}(.R) = \mathcal{D}^{1 \times p} / \mathcal{D}^{1 \times q} R$

→ $R \in \mathcal{D}^{q \times p}$, \mathcal{D} : a ring of PD operators

→ $u \in \mathcal{F}^p$, \mathcal{F} : a functional space (left \mathcal{D} -module)

→ $\mathcal{M} \leftrightarrow \mathcal{O} := \mathbb{K}[x_1, \dots, x_n] / I(f_1, \dots, f_r)$ from algebraic geometry

Malgrange's iso.: $\text{hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{F}) \simeq \ker_{\mathcal{F}}(R.)$ (i.e., \mathcal{F} -points \leftrightarrow \mathcal{F} -solutions)

→ investigate $(\Sigma : Ru = 0)$ by means of properties of \mathcal{M}

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Example: module of a system, Malgrange's iso.

System: 2 unknowns functions $u^1, u^2 \in \mathcal{F}$ and 3 equations

$$\begin{cases} \partial_1 u^1 =_{\mathcal{F}} 0 \\ \frac{1}{2}(\partial_2 u^1 + \partial_1 u^2) =_{\mathcal{F}} 0 \\ \partial_2 u^2 =_{\mathcal{F}} 0 \end{cases} \iff Ru = \begin{pmatrix} \partial_1 & 0 \\ \frac{1}{2}\partial_2 & \frac{1}{2}\partial_1 \\ 0 & \partial_2 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} =_{\mathcal{F}} 0$$

Corresponding module: $\mathcal{D} := \mathbb{Q}[\partial_1, \partial_2]$ and $\mathcal{M} := \mathcal{D}^{1 \times 2} / \mathcal{D}^{1 \times 3} R$

→ \mathcal{M} has 2 generators m_1, m_2 subject to 3 relations

$$\partial_1 \cdot m_1 =_{\mathcal{M}} 0 \quad \frac{1}{2} \partial_2 \cdot m_1 + \frac{1}{2} \partial_1 \cdot m_2 =_{\mathcal{M}} 0 \quad \partial_2 \cdot m_2 =_{\mathcal{M}} 0$$

Malgrange's iso. illustration: given $\varphi : \mathcal{M} \rightarrow \mathcal{F}$ and letting $u^j := \varphi(m_j)$, then $Ru =_{\mathcal{F}} 0$:

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Formal theory: solutions are formal power series

Procedure for constructing solutions:

- find relations between the coefficients of the Taylor development
→ candidate for the general solution
- formulate effective criteria for checking validity of the candidate
- deduce the general solution and the dimension of the solution space

Geometric and algebraic methods/tools

Differential geometry

Jet bundles, differential forms

- Σ_d is a submanifold of $J_d \rightarrow X$
 $(x^i, u^j_\mu) \rightarrow$ coord. of X and $(J_q)_x$
- differentiation of Σ_d
→ prolongation $J_d \rightarrow J_{d+1}$
- filtration of $\Sigma_d \rightarrow X \xrightarrow{\omega} J_\bullet \otimes \Lambda^\bullet T^*$

Differential algebra

Differential rings, (co)homology

- differential elimination
→ express $u^j_\mu =$ lower terms
→ compute integrability conditions
- homological algebra
→ validate integrability conditions

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Formal integrability

Def.: A system Σ_d (of degree d) is **formally integrable** if $\forall m, r \geq 0$:

$$\Sigma_{m+r+d} := J_{m+r}(\Sigma) \cap J_{m+r+d} \rightarrow \Sigma_{m+d} := J_m(\Sigma) \cap J_{m+d} \quad (+ \text{ regularity conditions})$$

Intuition: differentiations of degrees $m + r$ of F^ρ 's

$$\frac{\partial^{|\mu|} F^\rho}{\partial x^\mu}$$

do not bring new integrability conditions of degrees $\leq m + d$

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Example with one independent variable

$$\Sigma_1 = (\dot{u} = u) \subseteq J_1 \simeq \mathbb{R}^3 \rightarrow X \simeq \mathbb{R}, (t, u, \dot{u}) \mapsto t$$

Differential elimination: $u^{(m+r)} \rightarrow u^{(m+r-1)} \rightarrow u^{(m+1)} \rightarrow u^{(m)} \rightarrow \dots \rightarrow \dot{u} \rightarrow u \quad \forall m, r \geq 0$

New integrability conditions: involve $u^{(m+1)}, \dots, u^{(m+r)} \rightarrow \Sigma_1$ is formally integrable

Solution space: 1-dimensional space generated by e^{t-t_0} ($\Sigma_\infty := \lim(\dots \rightarrow \Sigma_2 \rightarrow \Sigma_1 \rightarrow \Sigma)$)

$$\Sigma_\infty \ni (t_0, u, u, u, \dots) \leftrightarrow u + u(t-t_0) + \frac{u}{2}(t-t_0)^2 + \frac{u}{3!}(t-t_0)^3 + \dots$$

Example with three independent variables

$$\Sigma_2 = (u_{33} = u_{23} = u_{13}, u_{12} = u_{11}) \subseteq J_2 \rightarrow X \simeq \mathbb{R}^3, (x^i, u, u_i, u_{ij}) \mapsto (x^i)$$

Differential elimination: using the lex order induced by $x^3 \succ x^2 \succ x^1$:

$$u_{233} \rightarrow u_{123} \rightarrow u_{113} \quad \text{and} \quad u_{233} \rightarrow u_{133} \rightarrow u_{113}$$

New integrability conditions: involve derivatives of degree $m + r + d$

→ Σ_2 is formally integrable

Solution space: generated by FPSs obtained from "parametric derivatives":

$$\begin{aligned} u(x^i) &= u + u_1 y^1 + u_2 y^2 + u_3 y^3 & (y^i := (x^i - x_0^i)) \\ &+ \frac{u_{11}}{2} (y^1)^2 + u_{11} y^1 y^2 + u_{13} y^1 y^3 + \frac{u_{22}}{2} (y^2)^2 + u_{13} y^2 y^3 + \frac{u_{13}}{2} (y^3)^2 \\ &+ \dots \end{aligned}$$

→ corresponds to $(x_0^i, u_i, u_{11}, u_{11}, u_{13}, u_{22}, u_{13}, u_{13}, \dots) \in \Sigma_\infty$

Janet example (not formally integrable)

$$\Sigma_2 = (u_{33} = x^2 u_{11}, u_{22} = 0) \subseteq J_2 \rightarrow X \simeq \mathbb{R}^3$$

Differential elimination: $u_{2233} \rightarrow 0$ and (D : formal derivative)

$$u_{2233} \rightarrow \frac{D}{DX^2} \left(\frac{D}{DX^2} (x^2 u_{11}) \right) = \frac{D}{DX^2} (x^2 u_{112} + u_{11}) = x^2 u_{1122} + 2u_{112} \rightarrow 2u_{112}$$

New integrability condition: $u_{112} = 0$, but u_{112} is free in Σ_3

$\rightarrow \Sigma_4 \rightarrow \Sigma_3$ is not onto: Σ_2 is not formally integrable

Solution space: ???

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$\rightarrow \Sigma_4 \rightarrow \Sigma_3$ **is not onto:** Σ_2 is not formally integrable

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Integrability defect and homology

Question: what type of finite test for proving formal integrability?

→ based on differential elimination (**S-polynomial like criterion**)

Fact: H^2 Spencer cohomology of the (comodule) symbol does not vanish for the Janet example

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Theorem (Goldschmidt). Assume that **the symbol of Σ_d is 2-acyclic**. If Σ_{d+1} is a fibered manifold and if $\Sigma_{d+1} \rightarrow \Sigma_d$ is onto, then Σ_d is formally integrable.

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Theorem (Serre). Spencer and Koszul complexes are dual to each other.

→ 2-acyclicity $\Leftrightarrow H_2$ **Koszul homology** vanishes

Proving formal integrability

Procedure:

- show that Σ_{d+1} is a fibered manifold \rightarrow rank computation
- proving 2-acyclicity \rightarrow **how?**
- show that $\Sigma_{d+1} \rightarrow \Sigma_d$ is onto \rightarrow differential elimination

If fail: apply a finite prolongation/projection method

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Our contribution:

compute H_2 Koszul homology of linear systems

\rightarrow using ORE Morphisms and ORE Modules Maple packages

II. COMPUTATION OF KOSZUL HOMOLOGY

Symbol module

Background: $(\Sigma_d : Ru = 0)$, $R \in \mathcal{D}^{q \times p}$, $\mathcal{D} = \mathcal{B}\langle \partial_1, \dots, \partial_n \rangle$

- \mathcal{B} : a commutative differential ring, equipped with derivations δ_i 's
- \mathcal{D} : generated by \mathcal{B} and ∂_i 's, subject to relations $\partial_i \partial_j = \partial_j \partial_i$, $\partial_i b = b \partial_i + \delta_i(b)$
- \mathcal{D} is a filtered ring: $\partial_i b = b \partial_i + \delta_i(b)$ is not homogeneous

Informal def.: the symbol module is the module of the top-degree part of Σ_d

- cancels low derivatives in the equations

Construction: based on the graded ring of $\mathcal{D} = \bigcup \mathcal{D}_m := \bigcup \mathcal{B}\langle \partial_1, \dots, \partial_n \rangle_m$

- $\mathcal{A} := \text{gr}(\mathcal{D}) = \bigoplus (\mathcal{D}_m / \mathcal{D}_{m-1})$, $\sigma_m : \mathcal{D}_m \rightarrow \mathcal{D}_m / \mathcal{D}_{m-1}$, $\chi_i := \sigma_1(\partial_i)$
- $\mathcal{A} \simeq \mathcal{B}[\chi_1, \dots, \chi_n]$: $\chi_i b = \sigma_1(\partial_i b) = \sigma_1(b \partial_i + \delta_i(b)) = \sigma_1(b \partial_i) = b \chi_i$
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Examples (with $p = 1$ unknown function)

Constant coefficients: $(\Sigma_2 : u_{33} = u_{23} = u_{13}, u_{12} = u_{11})$

- $R = (\partial_3^2 - \partial_1\partial_3 \quad \partial_2\partial_3 - \partial_1\partial_3 \quad \partial_1\partial_2 - \partial_1^2)^T$

- $\mathcal{D} = \mathbb{Q}[\partial_1, \partial_2, \partial_3] \simeq \mathcal{A} = \mathbb{Q}[\chi_1, \chi_2, \chi_3]$

($\mathcal{B} = \mathbb{Q}$ has trivial derivations: $\partial_i q = q\partial_i$)

→ $\text{gr}(\mathcal{M}) = \mathcal{A}/\mathcal{A}^{1 \times 3} (\chi_3^2 - \chi_1\chi_3 \quad \chi_2\chi_3 - \chi_1\chi_3 \quad \chi_1\chi_2 - \chi_1^2)^T$

Parametric coefficients: $(\Sigma_2 : u_{33} = u_{22} = u_{13} = u_{12} = 0, u_{23} = \alpha u_{11})$

- $R = (\partial_3^2 \quad \partial_2\partial_3 - \alpha\partial_1^2 \quad \partial_2^2 \quad \partial_1\partial_3 \quad \partial_1\partial_2)^T$

- $\mathcal{D} = \mathbb{Q}[\alpha][\partial_1, \partial_2, \partial_3] \simeq \mathcal{A} = \mathbb{Q}[\alpha][\chi_1, \chi_2, \chi_3]$

($\mathcal{B} = \mathbb{Q}[\alpha]$ has trivial derivations: $\partial_i p(\alpha) = p(\alpha)\partial_i$)

→ $\text{gr}(\mathcal{M}) = \mathcal{A}/\mathcal{A}^{1 \times 5} (\chi_3^2 \quad \chi_2\chi_3 - \alpha\chi_1^2 \quad \chi_2^2 \quad \chi_1\chi_3 \quad \chi_1\chi_2)^T$

Janet example

Polynomial coefficients: $(\Sigma_2 : u_{33} = x^2 u_{11}, u_{22} = 0)$

- $R = (\partial_3^2 - x^2 \partial_1^2 \quad \partial_2^2)^T$
 - $\mathcal{D} = \mathbb{A}_3 := \mathbb{Q}[x^1, x^2, x^3] \langle \partial_1, \partial_2, \partial_3 \rangle$: polynomial Weyl algebra
 $\mathbb{Q}[x^1, x^2, x^3]$ with usual derivations: $\partial_i x^j = x^j \partial_i + \delta_i^j$
 - $\mathcal{A} = \mathbb{Q}[x^1, x^2, x^3][\chi_1, \chi_2, \chi_3]$
- $\rightarrow \text{gr}(\mathcal{M}) = \mathcal{A}/\mathcal{A}^{1 \times 2} (\chi_3^2 - x^2 \chi_1^2 \quad \chi_2^2)^T$

Remark: $\mathcal{D} \not\cong \mathcal{A}$ as rings but $\mathcal{D} \simeq \mathcal{A}$ as vector spaces

Koszul complex

Let \mathcal{M} be a graded $\mathcal{A} := \mathcal{B}[\chi_1, \dots, \chi_n]$ -module (\mathcal{B} : a commutative ring)

Tangent module: $T := \mathcal{A}_1 = \mathcal{B}\chi_1 \oplus \dots \oplus \mathcal{B}\chi_n \simeq \mathcal{B}^n$

Exterior algebra: $\Lambda^\bullet T := \bigoplus \Lambda^k T = \mathcal{B} \oplus T \oplus \Lambda^2 T \oplus \dots \oplus \Lambda^n T$

→ \mathcal{B} -module generated by χ_1, \dots, χ_n subject to $\chi_i \wedge \chi_j = -\chi_j \wedge \chi_i$
(in particular: $\chi_i \wedge \chi_i = 0$)

→ free \mathcal{B} -module with basis $\chi_{i_1} \wedge \dots \wedge \chi_{i_k}$, $i_1 < \dots < i_k$

Koszul differential: $\partial_{k+1} : \Lambda^{k+1} T \otimes \mathcal{M} \rightarrow \Lambda^k T \otimes \mathcal{M}$ defined by

$$\partial_{k+1} (\chi_{i_1} \wedge \dots \wedge \chi_{i_{k+1}} \otimes m) = \sum_{j=1}^{k+1} (-1)^{j+1} \chi_{i_1} \wedge \dots \wedge \widehat{\chi_{i_j}} \wedge \dots \wedge \chi_{i_{k+1}} \otimes \chi_{i_j} m$$

Remark: ∂_k is \mathcal{A} -linear: $\partial_k(p(\mathbf{v} \otimes m)) = \partial_k(\mathbf{v} \otimes pm) = p\partial_k(\mathbf{v} \otimes m)$

Koszul complex II

Let \mathcal{M} be a graded $\mathcal{A} := \mathcal{B}[\chi_1, \dots, \chi_n]$ -module

Prop.: the Koszul complex $(\Lambda^\bullet T, \partial_\bullet)$ is a complex of \mathcal{A} -modules

Proof: $\partial_k \circ \partial_{k+1} (\chi_{i_1} \wedge \dots \wedge \chi_{i_{k+1}} \otimes m)$ is equal to

$$\begin{aligned} & \partial_k \left(\sum_{j=1}^{k+1} (-1)^{j+1} \chi_{i_1} \wedge \dots \wedge \widehat{\chi_{i_j}} \wedge \dots \wedge \chi_{i_{k+1}} \otimes \chi_{i_j} m \right) \\ &= \sum_{j=1}^{k+1} \sum_{l=1}^{j-1} (-1)^{j+l} \chi_{i_1} \wedge \dots \wedge \widehat{\chi_{i_l}} \wedge \dots \wedge \widehat{\chi_{i_j}} \wedge \dots \wedge \chi_{i_{k+1}} \otimes \chi_{i_l} \chi_{i_j} m \\ &+ \sum_{j=1}^{k+1} \sum_{l=j+1}^{k+1} (-1)^{j+l+1} \chi_{i_1} \wedge \dots \wedge \widehat{\chi_{i_l}} \wedge \dots \wedge \widehat{\chi_{i_j}} \wedge \dots \wedge \chi_{i_{k+1}} \otimes \chi_{i_l} \chi_{i_j} m \\ &= 0 \end{aligned}$$

2-acyclicity and involution

Let $(\Sigma_d : Ru = 0)$ and $\text{gr}(\mathcal{M})$ be the symbol module of Σ_d

$(R \in \mathcal{D}^{q \times p}, \mathcal{D} = \mathcal{B}\langle \partial_1, \dots, \partial_n \rangle, \mathcal{A} := \text{gr}(\mathcal{D}) = \mathcal{B}[\chi_1, \dots, \chi_n])$

Koszul homology of $\text{gr}(\mathcal{M})$: the Koszul complex (of \mathcal{A} -modules) is

$$0 \rightarrow \Lambda^n T \otimes \text{gr}(\mathcal{M}) \xrightarrow{\partial_n} \Lambda^{n-1} T \otimes \text{gr}(\mathcal{M}) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} T \otimes \text{gr}(\mathcal{M}) \xrightarrow{\partial_1} \text{gr}(\mathcal{M}) \rightarrow 0$$

The **Koszul homology** of $\text{gr}(\mathcal{M})$ is the \mathcal{A} -module $(\partial_{n+1} := 0 \text{ and } \partial_0 := 0)$

$$H_\bullet(\text{gr}(\mathcal{M})) := \bigoplus_{k=0}^n H_k(\text{gr}(\mathcal{M})) := \bigoplus_{k=0}^n \frac{\ker(\partial_k)}{\text{im}(\partial_{k+1})}$$

Definition. The symbol module $\text{gr}(\mathcal{M})$ of Σ_d is said to be

→ 2-acyclic if $H_2(\text{gr}(\mathcal{M})) = 0$

→ involutive if $H_1(\text{gr}(\mathcal{M})) = \dots = H_n(\text{gr}(\mathcal{M})) = 0$

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Computation of homology modules

Let $\mathcal{M}_i = \mathcal{A}^{1 \times p_i} / (\mathcal{A}^{1 \times q_i} R_i)$ and $\mathcal{M}_2 \xrightarrow{f} \mathcal{M}_1 \xrightarrow{g} \mathcal{M}_0$ be a complex of \mathcal{A} -modules

Fact: $\exists P_i$'s and Q_i 's such that the following diagram commutes

$$\begin{array}{ccccccc}
 \mathcal{A}^{1 \times q_2} & \xrightarrow{\cdot R_2} & \mathcal{A}^{1 \times p_2} & \longrightarrow & \mathcal{M}_2 & \longrightarrow & 0 \\
 \cdot Q_2 \downarrow & & \cdot P_2 \downarrow & & f \downarrow & & \\
 \mathcal{A}^{1 \times q_1} & \xrightarrow{\cdot R_1} & \mathcal{A}^{1 \times p_1} & \longrightarrow & \mathcal{M}_1 & \longrightarrow & 0 \\
 \cdot Q_1 \downarrow & & \cdot P_1 \downarrow & & g \downarrow & & \\
 \mathcal{A}^{1 \times q_0} & \xrightarrow{\cdot R_0} & \mathcal{A}^{1 \times p_0} & \longrightarrow & \mathcal{M}_0 & \longrightarrow & 0
 \end{array}$$

- generators of \mathcal{M}_i are mapped to lin. comb. of generators of $\mathcal{M}_{i-1} \rightarrow \exists P_i$
- generating relations of \mathcal{M}_i are mapped to 0 in $\mathcal{M}_{i-1} \rightarrow \exists Q_i$ st $R_i P_i = Q_i R_{i-1}$

Presentation of H : $\exists S', S'' : \ker \begin{pmatrix} \cdot P_1 \\ \cdot R_0 \end{pmatrix} = \mathcal{A}^{1 \times s'} \begin{pmatrix} S' & -S'' \end{pmatrix}$ and

$$\ker(g) / \text{im}(f) = \mathcal{A}^{1 \times s'} S' / \left(\mathcal{A}^{1 \times (p_2 + q_1)} \begin{pmatrix} P_2 \\ R_1 \end{pmatrix} \right)$$

Implementation: if \mathcal{A} is an Ore algebra, OreModules computes S' and S''

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Implementation: if \mathcal{A} is an Ore algebra, OreModules computes S' and S''

How to prove that homology vanishes?

Let $\mathcal{M}_2 \xrightarrow{f} \mathcal{M}_1 \xrightarrow{g} \mathcal{M}_0$ be a complex of \mathcal{A} -modules

Homology module: $H(\mathcal{M}_2 \rightarrow \mathcal{M}_1 \rightarrow \mathcal{M}_0) = \mathcal{A}^{1 \times s'} S' / \mathcal{A}^{1 \times s} S$

- lines of S' represent generators of $\ker(g)$
- lines of S represent generators of $\text{im}(f)$

Acyclicity criterion: the following are equivalent

- $H(\mathcal{M}_2 \xrightarrow{f} \mathcal{M}_1 \xrightarrow{g} \mathcal{M}_0) = 0$
- $\mathcal{A}^{1 \times s'} S' \subseteq \mathcal{A}^{1 \times s} S$
- there exists $T \in \mathcal{A}^{s' \times s}$ such that $S' = TS$

Implementation: if \mathcal{A} is an Ore algebra, `OreModules` computes T (if it exists)

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Computation of Koszul homology

$\text{gr}(\mathcal{M}) := \mathcal{A}^{1 \times p} / \mathcal{A}^{1 \times q} \sigma_d(R)$: symbol module of $(\Sigma_d : Ru = 0)$

Presentation of $\Lambda^k T \otimes \text{gr}(\mathcal{M})$: $\mathcal{A}^{1 \times r_k p} / \mathcal{A}^{1 \times r_k q} \sigma_d(R)$, $r_k := \dim_{\mathcal{B}}(\Lambda^k T)$

Matrix representation of ∂ :

$$\begin{array}{ccccc}
 \mathcal{A}^{1 \times r_{k+1} q} \xrightarrow{.I_{r_{k+1}} \otimes \sigma_d(R)} & \mathcal{A}^{1 \times r_{k+1} p} & \longrightarrow & \Lambda^{k+1} T \otimes \text{gr}(\mathcal{M}) & \longrightarrow 0 \\
 \downarrow \cdot P_{k+1} \otimes I_q & \downarrow \cdot P_{k+1} \otimes I_p & & \downarrow \partial_{k+1} & \\
 \mathcal{A}^{1 \times r_k q} \xrightarrow{.I_{r_k} \otimes \sigma_d(R)} & \mathcal{A}^{1 \times r_k p} & \longrightarrow & \Lambda^k T \otimes \text{gr}(\mathcal{M}) & \longrightarrow 0 \\
 \downarrow \cdot P_k \otimes I_q & \downarrow \cdot P_k \otimes I_p & & \downarrow \partial_k & \\
 \mathcal{A}^{1 \times r_{k-1} q} \xrightarrow{.I_{r_{k-1}} \otimes \sigma_d(R)} & \mathcal{A}^{1 \times r_{k-1} p} & \longrightarrow & \Lambda^{k-1} T \otimes \text{gr}(\mathcal{M}) & \longrightarrow 0
 \end{array}$$

$\rightarrow P_k$: analogous to grad, rot, div

Consequence: 2-acyclicity may be proven using OreMorphisms and OreModules

Remark: involution may be proven analogously

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 \mathcal{A}^{1 \times r_k q} \xrightarrow{.I_{r_k} \otimes \sigma_d(R)} \mathcal{A}^{1 \times r_k p} & \longrightarrow & \Lambda^k T \otimes \text{gr}(\mathcal{M}) & \longrightarrow & 0 \\
 \downarrow \cdot P_k \otimes I_q & & \downarrow \cdot P_k \otimes I_p & & \downarrow \partial_k \\
 \mathcal{A}^{1 \times r_{k-1} q} \xrightarrow{.I_{r_{k-1}} \otimes \sigma_d(R)} \mathcal{A}^{1 \times r_{k-1} p} & \longrightarrow & \Lambda^{k-1} T \otimes \text{gr}(\mathcal{M}) & \longrightarrow & 0
 \end{array}$$

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Consequence: 2-acyclicity may be proven using OreMorphisms and OreModules

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Theorem (C., Cluzeau, Quadrat; 2022).

Let Σ_d be a linear system of PDEs and let $\text{gr}(\mathcal{M})$ be the symbol module of Σ_d .
Let S, S' be two matrices such that

$$H_2(\text{gr}(\mathcal{M})) = \mathcal{A}^{1 \times s'} S' / (\mathcal{A}^{1 \times s} S)$$

Then, $\text{gr}(\mathcal{M})$ is 2-acyclic iff the top degree part of S' is a left-multiple of S .

III. Examples

Example 1

$$(\Sigma_2 : u_{33} = u_{13}, u_{23} = u_{13}, u_{12} = u_{11}), \quad \mathcal{A} = \mathbb{Q}[\chi_1, \chi_2, \chi_3]$$

$$\text{Symbol module: } \text{gr}(\mathcal{M}) = \mathcal{A}/\mathcal{A}^{1 \times 3} (\chi_3^2 - \chi_1\chi_3 \quad \chi_2\chi_3 - \chi_1\chi_3 \quad \chi_1\chi_2 - \chi_1^2)^T$$

$$\text{Koszul complex (degree 2): } \Lambda^3 T \otimes \text{gr}(\mathcal{M}) \xrightarrow{\partial_3} \Lambda^2 T \otimes \text{gr}(\mathcal{M}) \xrightarrow{\partial_2} T \otimes \text{gr}(\mathcal{M})$$

$$\rightarrow \partial_3(\chi_1 \wedge \chi_2 \wedge \chi_3 \otimes m) = \chi_2 \wedge \chi_3 \otimes \chi_1 m - \chi_1 \wedge \chi_3 \otimes \chi_2 m + \chi_1 \wedge \chi_3 \otimes \chi_3 m$$

$$\rightarrow \partial_2(\chi_i \wedge \chi_j \otimes m) = \chi_j \otimes \chi_i m - \chi_i \otimes \chi_j m.$$

$$\text{Ex. of 2-cycle: } \mathbf{c} := \chi_3 \wedge \chi_1 \otimes (\chi_1\chi_2 - \chi_2^2)$$

$$\rightarrow \partial_2(\mathbf{c}) = \chi_1 \otimes \underbrace{(\chi_1\chi_2\chi_3 - \chi_2^2\chi_3)}_{-\chi_2(\chi_2\chi_3 - \chi_1\chi_3)} + \chi_3 \otimes \underbrace{(-\chi_1^2\chi_2 + \chi_1\chi_2^2)}_{\chi_2(\chi_1\chi_2 - \chi_1^2)} = 0$$

$$\text{Ex. of 2-boundary: } \mathbf{c} = \partial_3(\mathbf{v}), \quad \mathbf{v} := \chi_1 \wedge \chi_2 \wedge \chi_3 \otimes (\chi_1 - \chi_2); \text{ indeed}$$

$$\partial_3(\mathbf{v}) = \chi_2 \wedge \chi_3 \otimes \underbrace{(\chi_1^2 - \chi_1\chi_2)}_0 + \underbrace{\chi_3 \wedge \chi_1 \otimes (\chi_1\chi_2 - \chi_2^2)}_{\mathbf{c}} + \chi_1 \wedge \chi_2 \otimes \underbrace{(\chi_1\chi_3 - \chi_2\chi_3)}_0 = \mathbf{c}$$

Example 1 (continued)

$$\text{gr}(\mathcal{M}) = \mathcal{A}/\mathcal{A}^{1 \times 3} \left(\chi_3^2 - \chi_1 \chi_3 \quad \chi_2 \chi_3 - \chi_1 \chi_3 \quad \chi_1 \chi_2 - \chi_1^2 \right)^T$$

$$\text{Koszul complex (degree 2): } \Lambda^3 T \otimes \text{gr}(\mathcal{M}) \xrightarrow{\partial_3} \Lambda^2 T \otimes \text{gr}(\mathcal{M}) \xrightarrow{\partial_2} T \otimes \text{gr}(\mathcal{M})$$

→ the \mathcal{A} -module $\Lambda^2 T \otimes \text{gr}(\mathcal{M})$ is generated by $\{\chi_2 \wedge \chi_3 \otimes 1, \chi_3 \wedge \chi_1 \otimes 1, \chi_1 \wedge \chi_2 \otimes 1\}$

$$H_2 \text{ Koszul homology: } H_2(\text{gr}(\mathcal{M})) = \mathcal{A}^{1 \times 7} S' / (\mathcal{A}^{1 \times 10} S)$$

$$S' = \begin{pmatrix} \chi_3 & \chi_3 & \chi_3 \\ \chi_1 & \chi_2 & \chi_3 \\ 0 & \chi_1 - \chi_2 & 0 \\ 0 & \chi_2 \chi_3 - \chi_3^2 & 0 \\ 0 & 0 & \chi_2 \chi_3 - \chi_3^2 \\ 0 & 0 & \chi_1 \chi_3 - \chi_3^2 \\ 0 & 0 & \chi_1^2 - \chi_1 \chi_2 \end{pmatrix} \quad S = \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 \\ \chi_3^2 - \chi_1 \chi_3 & 0 & 0 \\ \chi_2 \chi_3 - \chi_1 \chi_3 & 0 & 0 \\ \chi_1 \chi_2 - \chi_1^2 & 0 & 0 \\ 0 & \chi_3^2 - \chi_1 \chi_3 & 0 \\ 0 & \chi_2 \chi_3 - \chi_1 \chi_3 & 0 \\ 0 & \chi_1 \chi_2 - \chi_1^2 & 0 \\ 0 & 0 & \chi_3^2 - \chi_1 \chi_3 \\ 0 & 0 & \chi_2 \chi_3 - \chi_1 \chi_3 \\ 0 & 0 & \chi_1 \chi_2 - \chi_1^2 \end{pmatrix}$$

Remarks.

→ columns of S and $S' \leftrightarrow \{\mathbf{u} := \chi_2 \wedge \chi_3 \otimes 1, \mathbf{v} := \chi_3 \wedge \chi_1 \otimes 1, \mathbf{w} := \chi_1 \wedge \chi_2 \otimes 1\}$

→ $\chi_3(\mathbf{u} + \mathbf{v} + \mathbf{w})$ and $(\chi_1 - \chi_2)\mathbf{v}$ are $\neq 0$ in $H_2(\text{gr}(\mathcal{M}))$, but have degrees $1 \neq \deg(S')$

2-acyclicity test: success

Example 2

$$(\Sigma_2 : u_{33} = u_{22} = u_{13} = u_{12} = 0, u_{23} = \alpha u_{11}), \quad \mathcal{A} = \mathbb{Q}[\alpha][\chi_1, \chi_2, \chi_3]$$

Symbol module: $\text{gr}(\mathcal{M})_\alpha = \mathcal{A}/\mathcal{A}^{1 \times 5} \begin{pmatrix} \chi_3^2 & \chi_2\chi_3 & -\alpha\chi_1^2 & \chi_2^2 & \chi_1\chi_3 & \chi_1\chi_2 \end{pmatrix}^T$

Koszul complex (degree 2): $\Lambda^3 T \otimes \text{gr}(\mathcal{M})_\alpha \xrightarrow{\partial_3} \Lambda^2 T \otimes \text{gr}(\mathcal{M})_\alpha \xrightarrow{\partial_2} T \otimes \text{gr}(\mathcal{M})_\alpha$

$$\rightarrow \partial_3(\chi_1 \wedge \chi_2 \wedge \chi_3 \otimes m) = \chi_2 \wedge \chi_3 \otimes \chi_1 m - \chi_1 \wedge \chi_3 \otimes \chi_2 m + \chi_1 \wedge \chi_2 \otimes \chi_3 m$$

$$\rightarrow \partial_2(\chi_i \wedge \chi_j \otimes m) = \chi_j \otimes \chi_i m - \chi_i \otimes \chi_j m.$$

Ex. of 2-cycle: $\mathbf{c} := \chi_3 \wedge \chi_1 \otimes \alpha\chi_1^2$

$$\rightarrow \partial_2(\mathbf{c}) = \chi_1 \otimes \underbrace{\alpha\chi_1^2\chi_3}_{\alpha\chi_1(\chi_1\chi_3)} + \chi_3 \otimes (-\alpha\chi_1^3) = \chi_3 \otimes \left(\underbrace{(\chi_1\chi_2\chi_3 - \alpha\chi_1^3)}_{\chi_1(\chi_2\chi_3 - \alpha\chi_1^2)} - \underbrace{\chi_1\chi_2\chi_3}_{\chi_3(\chi_1\chi_2)} \right) = 0$$

Ex. of 2-boundary: $\mathbf{c} = \partial_3(\mathbf{v})$, $\mathbf{v} := \chi_1 \wedge \chi_2 \wedge \chi_3 \otimes \chi_3$; indeed

$$\rightarrow \partial_3(\mathbf{v}) = \chi_2 \wedge \chi_3 \otimes \chi_1\chi_3 + \chi_3 \wedge \chi_1 \otimes \chi_2\chi_3 + \chi_1 \wedge \chi_2 \otimes \chi_3^2 = \mathbf{c}$$

Example 3 (Janet example)

$$(\Sigma_2 : u_{33} = x^2 u_{11}, u_{22} = 0), \quad \mathcal{A} := \mathbb{Q}[x^1, x^2, x^3][\chi_1, \chi_2, \chi_3]$$

Symbol module: $\text{gr}(\mathcal{M}) = \mathcal{A}/\mathcal{A}^{1 \times 2} \begin{pmatrix} \chi_3^2 - x^2 \chi_1^2 & \chi_2^2 \end{pmatrix}^T$

Notation: $\mathbf{u} := \chi_2 \wedge \chi_3$, $\mathbf{v} := \chi_3 \wedge \chi_1$, $\mathbf{w} := \chi_1 \wedge \chi_2$

→ generate $\Lambda^2 \otimes \text{gr}(\mathcal{M})$ as an \mathcal{A} -module

Homology defect: $\mathbf{c} := \mathbf{u} \otimes \chi_2 \chi_3 + \mathbf{w} \otimes x^2 \chi_1 \chi_2 = \chi_2 \wedge \chi_3 \otimes \chi_2 \chi_3 + \chi_1 \wedge \chi_2 \otimes x^2 \chi_1 \chi_2$

$$\rightarrow \partial_2(\mathbf{c}) = \chi_1 \otimes \underbrace{\left(-x^2 \chi_1 \chi_2^2 \right)}_{-x^2 \chi_1 (\chi_2^2)} + \chi_2 \otimes \underbrace{\left(-\chi_2 \chi_3^2 + x^2 \chi_1^2 \chi_2 \right)}_{-x^2 (\chi_3^2 - x^2 \chi_1^2)} + \chi_3 \otimes \underbrace{\left(\chi_2^2 \chi_3 \right)}_{x^3 (\chi_2^2)} = 0$$

Moreover, $\partial_3(\mathbf{x} := \chi_1 \wedge \chi_2 \wedge \chi_3 \otimes (P_1 \chi_1 + P_2 \chi_2 + P_3 \chi_3)) = \lambda \mathbf{u} + \mu \mathbf{v} + \nu \mathbf{w}$, where

$$\lambda = P_1 \chi_1^2 + P_2 \chi_1 \chi_2 + P_3 \chi_1 \chi_3, \quad \mu = \dots, \quad \nu = \dots$$

Hence, $\partial_3(\mathbf{x}) = \mathbf{c} \rightarrow P_1 \chi_1^2 + P_2 \chi_1 \chi_2 + P_3 \chi_1 \chi_3 = \chi_2 \chi_3$: IMPOSSIBLE!

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Symbol module: $\text{gr}(\mathcal{M}) = \mathcal{A}/\mathcal{A}^{1 \times 2} \begin{pmatrix} \chi_3^2 - x^2 \chi_1^2 & \chi_2^2 \end{pmatrix}^T$

Notation: $\mathbf{u} := \chi_2 \wedge \chi_3$, $\mathbf{v} := \chi_3 \wedge \chi_1$, $\mathbf{w} := \chi_1 \wedge \chi_2$

→ generate $\Lambda^2 \otimes \text{gr}(\mathcal{M})$ as an \mathcal{A} -module

Homology defect: $\mathbf{c} := \mathbf{u} \otimes \chi_2 \chi_3 + \mathbf{w} \otimes x^2 \chi_1 \chi_2 = \chi_2 \wedge \chi_3 \otimes \chi_2 \chi_3 + \chi_1 \wedge \chi_2 \otimes x^2 \chi_1 \chi_2$

$$\rightarrow \partial_2(\mathbf{c}) = \chi_1 \otimes \underbrace{\left(-x^2 \chi_1 \chi_2^2 \right)}_{-x^2 \chi_1 (\chi_2^2)} + \chi_2 \otimes \underbrace{\left(-\chi_2 \chi_3^2 + x^2 \chi_1^2 \chi_2 \right)}_{-x^2 (\chi_3^2 - x^2 \chi_1^2)} + \chi_3 \otimes \underbrace{\left(\chi_2^2 \chi_3 \right)}_{x^3 (\chi_2^2)} = 0$$

Moreover, $\partial_3(\mathbf{x} := \chi_1 \wedge \chi_2 \wedge \chi_3 \otimes (P_1 \chi_1 + P_2 \chi_2 + P_3 \chi_3)) = \lambda \mathbf{u} + \mu \mathbf{v} + \nu \mathbf{w}$, where

$$\lambda = P_1 \chi_1^2 + P_2 \chi_1 \chi_2 + P_3 \chi_1 \chi_3, \quad \mu = \dots, \quad \nu = \dots$$

Hence, $\partial_3(\mathbf{x}) = \mathbf{c} \rightarrow P_1 \chi_1^2 + P_2 \chi_1 \chi_2 + P_3 \chi_1 \chi_3 = \chi_2 \chi_3$: **IMPOSSIBLE!**

Example 3 (continued)

$$\text{gr}(\mathcal{M}) = \mathcal{A}/\mathcal{A}^{1 \times 2} \begin{pmatrix} \chi_3^2 - x^2 \chi_1^2 & \chi_2^2 \end{pmatrix}^T$$

$$\text{Koszul complex (degree 2): } \Lambda^3 T \otimes \text{gr}(\mathcal{M}) \xrightarrow{\partial_3} \Lambda^2 T \otimes \text{gr}(\mathcal{M}) \xrightarrow{\partial_1} T \otimes \text{gr}(\mathcal{M})$$

$$H_2 \text{ Koszul homology: } H_2(\text{gr}(\mathcal{M})) = \mathcal{A}^{1 \times 8} S' / (\mathcal{A}^{1 \times 7} S)$$

$$S' = \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 \\ \chi_2^2 & 0 & 0 \\ \chi_2 \chi_3 & 0 & x^2 \chi_1 \chi_2 \\ \chi_3^2 & x^2 \chi_1 \chi_2 & x^2 \chi_1 \chi_3 \\ 0 & \chi_2^2 & 0 \\ 0 & x^2 \chi_1^2 - \chi_3^2 & 0 \\ 0 & 0 & \chi_2^2 \\ 0 & 0 & x^2 \chi_1^2 - \chi_3^2 \end{pmatrix} \quad S = \begin{pmatrix} \chi_1 & \chi_2 & \chi_3 \\ \chi_3^2 - x^2 \chi_1^2 & 0 & 0 \\ \chi_2^2 & 0 & 0 \\ 0 & \chi_3^2 - x^2 \chi_1^2 & 0 \\ 0 & \chi_2^2 & 0 \\ 0 & 0 & \chi_3^2 - x^2 \chi_1^2 \\ 0 & 0 & \chi_2^2 \end{pmatrix}$$

2-acyclicity test: fail

IV. CONCLUSION AND PERSPECTIVES

Conclusion and perspectives

Summary of presented results

- new criterion for proving 2-acyclicity of linear PDEs systems
- effective test using OreMorphisms and OreModules packages

Remark. Involutivity can also be checked with OreMorphism and OreModules

Further works

- go further into the effective approach to Spencer cohomology (Koszul-Tate theory, Spencer sequences)
- applications to physics and control theory (elasticity theory, minimal realization, ...)
- investigate rewriting systems over jet bundles (differential elimination, homotopy of rewriting systems)

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THANK YOU FOR LISTENING!